

# SYMMETRY RESULTS FOR COOPERATIVE ELLIPTIC SYSTEMS IN UNBOUNDED DOMAINS

LUCIO DAMASCELLI, FRANCESCA GLADIALI, AND FILOMENA PACELLA

**ABSTRACT.** In this paper we prove symmetry results for classical solutions of semilinear cooperative elliptic systems in  $\mathbb{R}^N$ ,  $N \geq 2$  or in the exterior of a ball. We consider the case of fully coupled systems and nonlinearities which are either convex or have a convex derivative.

The solutions are shown to be foliated Schwarz symmetric if a bound on their Morse index holds. As a consequence of the symmetry results we also obtain some nonexistence theorems.

## 1. Introduction and statement of the results

In this paper we study symmetry properties of classical  $C^2$  solutions of a semilinear elliptic system of the type

$$(1.1) \quad -\Delta U = F(|x|, U) \quad \text{in } \Omega$$

where  $F = (f_1, \dots, f_m) : \overline{\Omega} \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$  is locally a  $C^{1,\alpha}$  function,  $m \geq 2$  and  $\Omega$  is either  $\mathbb{R}^N$  or the exterior of a ball, i.e.  $\Omega = \mathbb{R}^N \setminus B$  where  $B$  is the unit ball centered at the origin and  $N \geq 2$ . In the second case we also require the boundary conditions

$$(1.2) \quad U = 0 \quad \text{on } \partial\Omega.$$

When  $\Omega = \mathbb{R}^N$  some radial symmetry results for positive solutions of (1.1) have been obtained using the classical moving plane method under further assumptions on  $F$  and/or on the decay of the solutions at infinity ([4], [1]).

As far as we know there are no results in the case when the solution changes sign or the underlying domain is the exterior of a ball.

Here we use the approach introduced in [11], [13], [9] (see also [12]) in the scalar case, i.e. when (1.1) reduces to a single equation, to prove a partial symmetry result for solutions of (1.1) and (1.2) having low Morse index, assuming some convexity on the nonlinear term  $F(|x|, U)$ . This approach is not immediately applicable to the case of systems, as explained in [2]. However in [2] and [3] using some new ideas and, in particular, considering an auxiliary symmetric system, foliated Schwarz symmetry of solutions is proved in the case of rotationally invariant bounded domains, i.e. balls or annuli.

In passing from bounded to unbounded domains new difficulties arise, some of which are peculiar of the vectorial case and do not appear in the scalar case. Therefore, though our strategy is mostly based on combining the approaches of [9] (for the

---

2010 *Mathematics Subject Classification.* 35B06, 35B50, 35J47, 35G60.

*Key words and phrases.* Cooperative elliptic systems, Symmetry, Maximum Principle, Morse index.

Supported by PRIN-2009-WRJ3W7 grant.

scalar equations in unbounded domains) and of [2] and [3] (for systems in bounded domains), we need some new devices, in particular we derive some other sufficient conditions for foliated Schwarz symmetry (see Section 3).

To precisely state our results we need a few definitions.

**Definition 1.** *Let  $\Omega$  be a rotationally symmetric domain in  $\mathbb{R}^N$ ,  $N \geq 2$ . We say that a continuous vector valued function  $U = (u_1, \dots, u_m) : \Omega \rightarrow \mathbb{R}^m$  is foliated Schwarz symmetric if each component  $u_i$  is foliated Schwarz symmetric with respect to the same vector  $p \in \mathbb{R}^N$ . In other words there exists a vector  $p \in \mathbb{R}^N$ ,  $|p| = 1$ , such that  $U(x)$  depends only on  $r = |x|$  and  $\theta = \arccos\left(\frac{x}{|x|} \cdot p\right)$  and  $U$  is (componentwise) nonincreasing in  $\theta$ .*

Next we define the Morse index for a solution  $U$  of (1.1) and (1.2). To this aim we denote by  $Q_U(-, \Omega)$  the quadratic form corresponding to a solution, i.e.

$$(1.3) \quad Q_U(\psi, \Omega) := \int_{\Omega} \left[ \sum_{i=1}^m |\nabla \psi_i|^2 - \sum_{i,j=1}^m \frac{\partial f_i}{\partial u_j}(|x|, U) \psi_j \psi_i \right] dx$$

where  $\psi = (\psi_1, \dots, \psi_m) \in C_c^1(\Omega, \mathbb{R}^m)$ , i.e. is compactly supported in  $\Omega$ . Throughout the paper we will frequently use the fact that  $Q_U$  is also well defined on functions in  $H_0^1(\Omega, \mathbb{R}^m)$  which vanish outside a compact set.

**Definition 2.** *Let  $U$  be a  $C^2(\Omega, \mathbb{R}^m)$  solution of (1.1) and (1.2). We say that*

- *$U$  is linearized stable, or it has zero Morse index, if  $Q_U(\psi, \Omega) \geq 0$  for any  $\psi \in C_c^1(\Omega, \mathbb{R}^m)$ ;*
- *$U$  has (linearized) Morse index equal to the integer  $\mu = \mu(U) \geq 1$ , if  $\mu$  is the maximal dimension of a subspace  $X \subset C_c^1(\Omega, \mathbb{R}^m)$  such that  $Q_U(\psi, \Omega) < 0$  for any  $\psi \in X \setminus \{0\}$ .*

Finally we recall some coupling conditions for systems.

**Definition 3.** • *We say that the system (1.1) is cooperative or weakly coupled in an open set  $D \subseteq \Omega$  if*

$$\frac{\partial f_i}{\partial u_j}(|x|, u_1, \dots, u_m) \geq 0 \quad \text{for any } (x, u_1, \dots, u_m) \in D \times \mathbb{R}^m$$

*for any  $i, j = 1, \dots, m$  with  $i \neq j$ .*

- *We say that the system (1.1) is fully coupled along a solution  $U$  in an open set  $D \subseteq \Omega$  if it is cooperative in  $D$  and, in addition, for any  $I, J \subset \{1, \dots, m\}$  such that  $I \neq \emptyset$ ,  $J \neq \emptyset$ ,  $I \cap J = \emptyset$ ,  $I \cup J = \{1, \dots, m\}$  there exist  $i_0 \in I$ ,  $j_0 \in J$  such that*

$$\text{meas} \left( \left\{ x \in D : \frac{\partial f_{i_0}}{\partial u_{j_0}}(|x|, U(x)) > 0 \right\} \right) > 0.$$

Let  $e \in S^{N-1}$  be a direction, i.e.  $e \in \mathbb{R}^N$ ,  $|e| = 1$ , and let us define the set

$$(1.4) \quad \Omega(e) = \{x \in \Omega : x \cdot e > 0\}.$$

Our main symmetry results are contained in the following theorems

**Theorem 1.1.** *Let  $U$  be a solution of (1.1) and (1.2) such that  $|\nabla U| \in L^2(\Omega)$  and Morse index  $\mu(U) \leq N$ . Assume that:*

- i) *The system is fully coupled along  $U$  in  $\Omega(e)$  for any  $e \in S^{N-1}$ .*
- ii) *For any  $i, j = 1, \dots, m$   $\frac{\partial f_i}{\partial u_j}(|x|, u_1, \dots, u_m)$  is nondecreasing in each variable  $u_k$ ,  $k = 1, \dots, m$  for any  $x \in \Omega$ .*
- iii) *If  $m \geq 3$ , for any  $i \in \{1, \dots, m\}$ ,  $f_i(|x|, u_1, \dots, u_m) = \sum_{\substack{k=1 \\ k \neq i}}^m g_{ik}(|x|, u_i, u_k)$  where  $g_{ik} \in C^{1,\alpha}([0, +\infty) \times \mathbb{R}^2)$ .*

*Then  $U$  is foliated Schwarz symmetric.*

**Theorem 1.2.** *Let  $U$  be a solution of (1.1) and (1.2) such that  $|\nabla U| \in L^2(\Omega)$  and Morse index  $\mu(U) \leq N - 1$ . Assume that:*

- i) *The system is fully coupled along  $U$  in  $\Omega$ .*
- ii) *For any  $i, j = 1, \dots, m$  the function  $\frac{\partial f_i}{\partial u_j}(|x|, U)$  is convex in  $U$ :*

$$\frac{\partial f_i}{\partial u_j}(|x|, tU(x_1) + (1-t)U(x_2)) \leq t \frac{\partial f_i}{\partial u_j}(|x|, U(x_1)) + (1-t) \frac{\partial f_i}{\partial u_j}(|x|, U(x_2))$$

*for any  $x, x_1, x_2 \in \Omega$  and for any  $t \in [0, 1]$ .*

*Then  $U$  is foliated Schwarz symmetric.*

Theorem 1.1 extends the main result of [2] to unbounded radial domains, while Theorem 1.2 extends the symmetry result of [3] to the same unbounded domains.

Let us note that our results do not require solutions to be bounded and neither to belong to  $H_0^1(\Omega)$ , but only that  $|\nabla U| \in L^2(\Omega)$ .

The proofs of Theorem 1.1 and 1.2 are technically quite complicated. However we want to point out that a crucial point is to have suitable sufficient conditions for the Schwarz symmetry, namely the ones contained in Section 3, in particular Proposition 3.9.

Moreover, in the case of Theorem 1.1 and 1.2, to bypass the difficulty of dealing with a nonselfadjoint linearized operator we use, as in [2], the linear operator associated with the symmetric part of the jacobian matrix  $J_F$  of  $F$  which is selfadjoint and to which the same quadratic form (1.3) corresponds.

Note that if the system is of gradient type, i.e.  $F = \text{grad}(g)$ , for some scalar function  $g$  (see [5]) then the quadratic form corresponds to that generated by the second derivative of a suitable associated functional and hence the linearized operator is selfadjoint. However this is not the case for many interesting systems as the so-called “hamiltonian-systems” ([5], [2]).

The two above symmetry theorems can be applied to different kind of systems and solutions. In the first one, the hypothesis ii) which implies that each  $f_i$  is convex with respect to each variable  $u_j$ ,  $i, j = 1, \dots, m$ , seems, in some cases, suitable for positive solutions. Moreover Theorem 1.1 applies to solutions with Morse index up to the dimension  $N$  and, for  $m \geq 3$ , requires an additional hypothesis.

Instead Theorem 1.2 applies more generally to sign changing solutions and does not need extra-assumptions for  $m \geq 3$ . On the contrary the hypothesis on the Morse index is more restrictive since it requires  $\mu(U) \leq N - 1$ .

As a consequence of the proofs of the symmetry results we derive a necessary condition to be satisfied by a solution which could be used to establish some nonexistence results

**Theorem 1.3.** *Assume  $U$  is a nonradial solution of (1.1) and (1.2) and either*

- a)  *$U$  has Morse index one and satisfies the assumptions of Theorem 1.1 or of Theorem 1.2;*
- or
- b) *the assumptions of Theorem 1.2 are satisfied and there exist  $i_0, j_0 \in \{1, \dots, m\}$  such that  $\frac{\partial f_{i_0}}{\partial u_{j_0}}(|x|, S)$  satisfies the following strict convexity assumption:*

$$(1.5) \quad \frac{\partial f_{i_0}}{\partial u_{j_0}}(|x|, tS' + (1-t)S'') < t \frac{\partial f_{i_0}}{\partial u_{j_0}}(|x|, S') + (1-t) \frac{\partial f_{i_0}}{\partial u_{j_0}}(|x|, S'')$$

*for any  $t \in (0, 1)$ , whenever  $x \in \Omega$  and  $S' = (s'_1, \dots, s'_m)$ ,  $S'' = (s''_1, \dots, s''_m) \in \mathbb{R}^m$  satisfy  $s'_k \neq s''_k$  for any  $k \in \{1, \dots, m\}$ .*

*Then necessarily*

$$(1.6) \quad \sum_{j=1}^m \frac{\partial f_i}{\partial u_j}(r, U(r, \theta)) \frac{\partial u_j}{\partial \theta}(r, \theta) = \sum_{j=1}^m \frac{\partial f_j}{\partial u_i}(r, U(r, \theta)) \frac{\partial u_j}{\partial \theta}(r, \theta)$$

*for any  $i = 1, \dots, m$ , with  $(r, \theta)$  as in Definition 1. In particular, if  $m = 2$ , from (1.6) we derive that*

$$(1.7) \quad \frac{\partial f_1}{\partial u_2}(|x|, U(x)) = \frac{\partial f_2}{\partial u_1}(|x|, U(x)) \quad , \quad \text{for any } x \in \Omega .$$

The symmetry results of the previous theorems hold, in particular, for stable solutions of (1.1) and (1.2). However for these solutions (as in the scalar case) we easily obtain the radial symmetry without requiring any hypothesis on the nonlinearity. Moreover, if  $\Omega = \mathbb{R}^N$  in the autonomous case we get that stable solutions must be constant.

**Theorem 1.4.** *Every linearized stable solution of (1.1) and (1.2) such that  $|\nabla U| \in L^2(\Omega)$  is radial. If, in addition,  $\Omega = \mathbb{R}^N$  and  $F = F(U)$ , i.e.  $F$  does not depend on  $|x|$ , then  $U$  is constant.*

**Remark 1.5.** This result is analogous to the one for scalar equations obtained in [9]. We observe that our definition of linearized stability is stronger than the one used in [8] to get a nonexistence result for Hénon-Lane-Emden systems.

Finally, as corollary of the symmetry theorems we get some nonexistence results, analogous to those obtained in the scalar case (see [9]), but under the stronger assumptions of Theorems 1.3.

**Theorem 1.6.** *Assume that one among assumptions a) and b) of Theorem 1.3 holds. If  $\Omega = \mathbb{R}^N$  and  $F = F(U)$ , i.e.  $F$  does not depend on  $|x|$ , then there are no sign changing solutions of (1.1) such that*

$$U(x) \rightarrow 0 \quad \text{as } |x| \rightarrow +\infty.$$

**Theorem 1.7.** *Assume that one between assumptions a) and b) of Theorem 1.3 is satisfied. If  $\Omega = \mathbb{R}^N \setminus B$  and  $F = F(U)$ , i.e.  $F$  does not depend on  $|x|$ , then there are no positive solutions of (1.1) (1.2) such that*

$$U(x) \rightarrow 0 \quad \text{as } |x| \rightarrow +\infty.$$

For systems of gradient type the previous results can be improved in the sense that they hold under the assumptions of Theorem 1.1 or Theorem 1.2.

**Theorem 1.8.** *Assume that either the assumptions of Theorem 1.1 or of Theorem 1.2 hold. If the system (1.1) is of gradient type and  $F$  does not depend on  $|x|$ , then there are neither sign changing solutions of (1.1) in  $\mathbb{R}^N$  nor positive solutions of (1.1) and (1.2) in  $\mathbb{R}^N \setminus B$  such that*

$$U(x) \rightarrow 0 \quad \text{as } |x| \rightarrow +\infty.$$

We observe that the case of the system of gradient type is easier. Indeed for this kind of systems the existence of a positive solution of the linearized equation ensures, as in Lemma 2.1 of [9], the positivity of the quadratic form associated to it, and, in some sense, the validity of the maximum principle. This is not true anymore for systems which are not of gradient type. In the case of bounded domains (see [2] and [3]), to overcome this difficulty we used the principal eigenvalue which would not help for unbounded domains, while here we use a new sufficient condition for the foliated Schwarz symmetry.

We conclude with a few remarks on the range of applicability of our theorems.

All our results require some information on the Morse index of the solution. If the system is of gradient type, as recalled, the quadratic form corresponds to that generated by the second derivative of a suitable associated functional. So, often, variational methods, used to find solutions, also carry information on the Morse index (see [5]), (see [10], for an example). A standard case is given by solutions obtained by the Mountain Pass theorem.

If the system is not of this type it could happen that the second derivative of functionals associated to the variational formulation of the system are strongly indefinite.

As explained and showed in [2] this does not mean that solutions do not have finite (linearized) Morse index. We believe that, in general, more investigation about stability properties of solutions of systems should be done.

We also think that the result of Theorem 1.3 is interesting and could give some new understanding of systems which have or do not have solutions and are not of gradient type.

The outline of the paper is the following. In Section 2 we state or prove some preliminary results while in Section 3 we show sufficient conditions for the foliated Schwarz symmetry. In Section 4 we give the proofs of the two symmetry results Theorem 1.1 and Theorem 1.2. Finally Section 5 is devoted to the remaining theorems.

## 2. Notations and preliminary results

We fix some general notation. Throughout the paper,  $B_R$  denotes the ball in  $\mathbb{R}^N$  with radius  $R > 0$  centered at the origin. If  $D$  is a domain, we denote by  $C_c^1(D, \mathbb{R}^m)$  the space of all  $C^1$ -functions from  $D$  to  $\mathbb{R}^m$  compactly supported in  $D$ .

We start by recalling some statements about linear systems.

Let  $D$  be any smooth domain of  $\mathbb{R}^N$  and let  $A(x)$  be an  $m \times m$  matrix defined in  $D$ , i.e.  $A(x) = (a_{ij}(x))_{i,j=1}^m$  and  $a_{ij}(x) \in L_{loc}^\infty(D)$ . We consider the linear elliptic

system

$$\begin{cases} -\Delta\psi + A(x)\psi = 0 & \text{in } D \\ \psi = 0 & \text{on } \partial D \end{cases}$$

i.e the system

$$(2.8) \quad \begin{cases} -\Delta\psi_1 + \sum_{j=1}^m a_{1j}(x)\psi_j = 0 & \text{in } D \\ \dots\dots\dots \\ -\Delta\psi_m + \sum_{j=1}^m a_{mj}(x)\psi_j = 0 & \text{in } D \\ \psi_1 = \dots = \psi_m = 0 & \text{on } \partial D. \end{cases}$$

We say that a function  $\psi \in H_0^1(D, \mathbb{R}^m)$  is a weak solution of (2.8) if

$$\int_D \sum_{i=1}^m \nabla\psi_i \cdot \nabla\phi_i + \sum_{i,j=1}^m a_{ij}(x)\psi_j\phi_i \, dx = 0$$

for any  $\phi \in C_c^1(D, \mathbb{R}^m)$ . We will denote by  $(-, -)_{L^2}$  the scalar product in  $L^2(D, \mathbb{R}^m)$ , i.e.  $(\psi, \phi)_{L^2}^2 = \int_D \sum_{i=1}^m \psi_i\phi_i \, dx$  and by  $\nabla\psi \cdot \nabla\phi = \sum_{i=1}^m \nabla\psi_i \cdot \nabla\phi_i$  for any vector valued function  $\psi, \phi \in C^1(D, \mathbb{R}^m)$ . We recall the definition of weakly and fully coupled for this kind of systems.

**Definition 4.** *The linear system (2.8) is said to be*

- cooperative or weakly coupled in  $D$  if  $a_{ij}(x) \leq 0$  a.e. in  $D$  whenever  $i \neq j$ ;
- fully coupled in  $D$  if it is weakly coupled in  $D$  and for any  $I, J \subset \{1, \dots, m\}$  such that  $I, J \neq \emptyset$ ,  $I \cap J = \emptyset$ ,  $I \cup J = \{1, \dots, m\}$  there exists  $i_0 \in I$  and  $j_0 \in J$  such that  $\text{meas}(\{x \in D : a_{i_0 j_0}(x) < 0\}) > 0$ .

For any scalar function  $g$ , we set  $g^+ = \max\{g, 0\}$  and  $g^- = \min\{g, 0\}$ . Similarly, for any vector-valued function  $W = (w_1, \dots, w_m)$  we set  $W^+ = (w_1^+, \dots, w_m^+)$  and  $W^- = (w_1^-, \dots, w_m^-)$ . Here and in the sequel, inequalities involving vectors should be understood to hold componentwise, for example  $\psi = (\psi_1, \dots, \psi_m) \geq 0$  means  $\psi_i \geq 0$  for any index  $i = 1, \dots, m$ .

We recall some known facts about Maximum Principle for systems, see [5], [6], [15] and reference therein for more details.

**Theorem 2.1** (Strong Maximum Principle and Hopf's Lemma). *Suppose that the linear system (2.8) is fully coupled in  $D$  and  $U = (u_1, \dots, u_m) \in C^1(\overline{D}; \mathbb{R}^m)$  is a weak solution of (2.8). If  $U \geq 0$  in  $D$ , then either  $U \equiv 0$  in  $D$  or  $U > 0$  in  $D$ . In the latter case if  $P \in \partial D$  and  $U(P) = 0$  then  $\frac{\partial U}{\partial \nu}(P) < 0$ , where  $\nu$  is the unit exterior normal vector at  $P$ .*

**Definition 5.** *We say that the maximum principle holds for the operator  $-\Delta + A(x)$  in an open set  $D \subseteq \Omega$  if any  $U \in H^1(D, \mathbb{R}^m)$  such that  $U \leq 0$  on  $\partial D$  (i.e.  $U^+ \in H_0^1(D, \mathbb{R}^m)$ ) and such that  $-\Delta U + A(x)U \leq 0$  in  $D$  (i.e.  $\int_D \sum_{i=1}^m \nabla u_i \cdot \nabla \psi_i + \sum_{i,j=1}^m a_{ij}(x)u_j\psi_i \, dx \leq 0$  for any nonnegative  $\psi \in H_0^1(D, \mathbb{R}^m)$ ) satisfies  $U \leq 0$  a.e. in  $D$ .*

**Theorem 2.2.** *There exists  $\delta > 0$ , depending on  $A(x)$ , such that for any subdomain  $D \subseteq \Omega$  the maximum principle holds for the operator  $-\Delta + A(x)$  in  $D \subseteq \Omega$  provided  $\text{meas}(D) \leq \delta$ .*

We refer to [2] for a general formulation and a proof of Theorem 2.1 and Theorem 2.2.

Given the linear system (2.8) in  $D \subseteq \Omega$ , we can associate with it the quadratic form

$$(2.9) \quad Q_A(\psi, D) := \int_D \sum_{i=1}^m |\nabla \psi_i|^2 + \sum_{i,j=1}^m a_{ij}(x) \psi_i \psi_j dx$$

for any  $\psi \in C_c^1(D, \mathbb{R}^m)$ . By density, the quadratic form  $Q_A$  is well defined also if  $\psi$  is in  $H_0^1(D, \mathbb{R}^m)$  and vanishes a.e. outside a bounded set.

We now consider the (symmetric) linear system defined by:

$$(2.10) \quad \begin{cases} -\Delta \psi_1 + \frac{1}{2} \sum_{j=1}^m (a_{1j}(x) + a_{j1}(x)) \psi_j = 0 & \text{in } D \\ \dots\dots\dots \\ -\Delta \psi_m + \frac{1}{2} \sum_{j=1}^m (a_{mj}(x) + a_{jm}(x)) \psi_j = 0 & \text{in } D \\ \psi_1 = \dots = \psi_m = 0 & \text{on } \partial D, \end{cases}$$

and observe that to (2.8) and (2.10) corresponds the same quadratic form (2.9). Obviously if the matrix  $A(x)$  of (2.8) is symmetric then (2.8) and (2.10) coincide. Moreover we remark that if (2.8) is weakly (fully) coupled in  $D$  then the same holds for the symmetric system (2.10).

Next we consider the bilinear symmetric form associated to (2.9):

$$(2.11) \quad P_A(\psi, \phi, D) := \int_D \sum_{i=1}^m \nabla \psi_i \cdot \nabla \phi_i + \frac{1}{2} \sum_{i,j=1}^m (a_{ij}(x) + a_{ji}(x)) \psi_j \phi_i dx$$

for any  $\psi, \phi \in C_c^1(D, \mathbb{R}^m)$  (but also for  $\psi, \phi \in H_0^1(D, \mathbb{R}^m)$  which vanish outside a bounded set). Note that if the quadratic form (2.9) is positive semidefinite then the bilinear symmetric form (2.11) defines a scalar product and hence the Cauchy-Schwarz inequality holds, i.e.

$$(P_A(\psi, \phi, D))^2 \leq P_A(\psi, \psi, D) P_A(\phi, \phi, D) = Q_A(\psi, D) Q_A(\phi, D)$$

for any  $\psi, \phi \in H_0^1(D, \mathbb{R}^m)$  vanishing a.e. outside a bounded set.

If  $D$  is bounded then the symmetric system (2.10) has a sequence of Dirichlet eigenvalues  $\lambda_k$  such that  $\lambda_k \rightarrow +\infty$  as  $k \rightarrow +\infty$  and a corresponding sequence of eigenfunctions  $Z^k \in H_0^1(D, \mathbb{R}^m)$ , i.e. functions that satisfy

$$\begin{cases} -\Delta Z_i^k + \sum_{j=1}^m \frac{1}{2} (a_{ij}(x) + a_{ji}(x)) Z_j^k = \lambda_k Z_i^k & \text{in } D \\ Z^k = 0 & \text{on } \partial D. \end{cases}$$

We denote by  $\lambda_k^s(L_A, D)$  these symmetric eigenvalues.

The first symmetric eigenfunction, i.e. the function associated with  $\lambda_1^s(L_A, D)$  does not change sign in  $D$  and the first eigenvalue is simple, i.e. up to a scalar multiplication there is only one eigenfunction corresponding to  $\lambda_1^s(L_A, D)$ .

See [2] for a careful study of the properties of these symmetric eigenvalues.

**Lemma 2.3.** *Let  $L_A$  and  $Q_A$  be defined as above. If*

$$(2.12) \quad \inf_{\psi \in C_c^1(\Omega(e), \mathbb{R}^m)} Q_A(\psi, \Omega(e)) < 0 \quad \text{for any } e \in S^{N-1}$$

*with  $\Omega(e)$  as in (1.4), then there exists  $\tilde{R} > 0$  (depending on  $A(x)$ ) such that, for any  $e \in S^{N-1}$  and for any  $R \geq \tilde{R}$ , the first symmetric eigenvalue  $\lambda_1^s(L_A, \Omega(e) \cap B_R)$  of the operator  $L_A$  in the domain  $B_R \cap \Omega(e)$  with zero Dirichlet boundary conditions is negative.*

*Proof.* Arguing by contradiction we assume there exists a sequence of directions  $e_n \in S^{N-1}$  and a sequence of radii  $R_n \rightarrow +\infty$  such that

$$(2.13) \quad \lambda_1^s(L_A, \Omega(e_n) \cap B_{R_n}) \geq 0.$$

Up to a subsequence  $e_n \rightarrow \tilde{e} \in S^{N-1}$  and it holds

$$(2.14) \quad \inf_{\psi \in C_c^1(\Omega(\tilde{e}), \mathbb{R}^m)} Q_A(\psi, \Omega(\tilde{e})) \geq 0$$

contradicting (2.12). Indeed, if (2.14) does not hold there should exist a function  $\psi \in C_c^1(\Omega(\tilde{e}), \mathbb{R}^m)$  such that  $Q_A(\psi, \Omega(\tilde{e})) < 0$  and this would imply that  $\lambda_1^s(L_A, \Omega(\tilde{e}) \cap B_{R_n}) < 0$  for  $n$  sufficiently large. The continuity of the first symmetric eigenvalue implies that  $\lambda_1^s(L_A, \Omega(e_n) \cap B_{R_n}) < 0$  for  $n$  large enough contradicting (2.13).  $\square$

We introduce some more notation. For a unit vector  $e \in S^{N-1}$  we consider the hyperplane  $T(e) = \{x \in \mathbb{R}^N, : x \cdot e = 0\}$ . We write  $\sigma_e : \Omega \mapsto \Omega$  for the reflection with respect to  $T(e)$ , that is  $\sigma_e(x) = x - 2(x \cdot e)e$  for every  $x \in \Omega$ , and denote by  $U^{\sigma_e} = (u_1^{\sigma_e}, \dots, u_m^{\sigma_e})$  the function  $U \circ \sigma_e$ . Note that  $T(-e) = T(e)$  and  $\Omega(-e) = \sigma_e(\Omega(e)) = -\Omega(e)$  for every  $e \in S^{N-1}$ . Finally for a given direction  $e \in S^{N-1}$  let us denote by  $W^e = (w_1, \dots, w_m)$  the difference between  $U$  and its reflection with respect to the hyperplane  $T(e)$ , i.e.  $W^e(x) = U(x) - U(\sigma_e(x))$ . Obviously the function  $W^e$  satisfies the linear system  $-\Delta W^e = F(|x|, U) - F(|x|, U^{\sigma_e})$  in  $\Omega$  and in  $\Omega(e)$ . This system can be written as a linear system in many ways. Indeed we need at least two different formulations of it to deal with the different hypotheses of Theorem 1.1 and Theorem 1.2.

First, it is standard to see that the function  $W^e$  satisfies in  $\Omega(e)$  and in  $\Omega$  the system

$$(2.15) \quad \begin{cases} -\Delta W^e + B^e(x)W^e = 0 & \text{in } \Omega(e) \\ W^e = 0 & \text{on } \partial\Omega(e) \end{cases}$$

where  $B^e(x) = (b_{ij}^e(x))_{i,j=1}^m$  with

$$(2.16) \quad b_{ij}^e(x) = - \int_0^1 \frac{\partial f_i}{\partial u_j}(|x|, tU(x) + (1-t)U(\sigma_e(x))) dt.$$

Now, we can write

$$\begin{aligned} & f_i(|x|, U(x)) - f_i(|x|, U(\sigma_e(x))) \\ &= f_i(|x|, u_1(x), \dots, u_m(x)) - f_i(|x|, u_1^{\sigma_e}(x), u_2(x), \dots, u_m(x)) \\ &+ f_i(|x|, u_1^{\sigma_e}(x), u_2(x), \dots, u_m(x)) - f_i(|x|, u_1^{\sigma_e}(x), u_2^{\sigma_e}(x), \dots, u_m(x)) \\ &\dots \dots \\ &+ f_i(|x|, u_1^{\sigma_e}(x), \dots, u_{m-1}^{\sigma_e}(x), u_m(x)) - f_i(|x|, u_1^{\sigma_e}(x), \dots, u_m^{\sigma_e}(x)) \\ &= - \left( \tilde{b}_{i1}^e(x)w_1 + \dots + \tilde{b}_{im}^e(x)w_m \right) \end{aligned}$$

where

$$(2.17) \quad \tilde{b}_{ij}^e(x) = - \int_0^1 \frac{\partial f_i}{\partial u_j}(|x|, u_1^{\sigma_e}, \dots, u_{j-1}^{\sigma_e}, tu_j(x) + (1-t)u_j(\sigma_e(x)), u_{j+1}, \dots, u_m) dt.$$



This implies that the function  $W^e$  satisfies in  $\Omega(e)$  and in  $\Omega$  the linear system

$$(2.18) \quad \begin{cases} -\Delta W^e + \tilde{B}^e(x)W^e = 0 & \text{in } \Omega(e) \\ W^e = 0 & \text{on } \partial\Omega(e) \end{cases}$$

where  $\tilde{B}^e(x) = \left(\tilde{b}_{ij}^e(x)\right)_{i,j=1}^m$ .

We collect some properties of these linear systems in the following lemma.

**Lemma 2.4.** *Let  $U$  be a solution of (1.1) and (1.2) and  $e$  any direction,  $e \in S^{N-1}$ .*

i) *Assume that hypotheses i), ii) and iii) of Theorem 1.1 hold. Then, for any  $x \in \Omega$ , we have*

$$(2.19) \quad \tilde{b}_{ii}^e(x) \geq -\frac{\partial f_i}{\partial u_i}(|x|, U(x)) \quad \text{if } u_i(x) \geq u_i^{\sigma_e}(x),$$

$$(2.20) \quad \tilde{b}_{ij}^e(x) \geq -\frac{\partial f_i}{\partial u_j}(|x|, U(x)) \quad \text{if } u_i(x) \geq u_i^{\sigma_e}(x), u_j(x) \geq u_j^{\sigma_e}(x)$$

*in particular, if  $u_i(x) = u_i^{\sigma_e}(x)$ ,  $u_j(x) = u_j^{\sigma_e}(x)$  then  $\tilde{b}_{ij}^e(x) = -\frac{\partial f_i}{\partial u_j}(|x|, U(x))$ .*

*Moreover the system (2.18) is fully coupled in  $\Omega$  and in  $\Omega(e)$ .*

ii) *If the system (1.1) is fully coupled along  $U$  in  $\Omega$  then the linear system (2.15) is fully coupled in  $\Omega$  and in  $\Omega(e)$  for any  $e \in S^{N-1}$ . If also hypothesis ii) of Theorem 1.2 holds, and we let, for any direction  $e \in S^{N-1}$*

$$(2.21) \quad b_{ij}^{e,s}(x) = \frac{1}{2} \left( \frac{\partial f_i}{\partial u_j}(|x|, U(x)) + \frac{\partial f_i}{\partial u_j}(|x|, U(\sigma_e(x))) \right)$$

*then, for any  $i, j = 1, \dots, m$  and  $x \in \Omega$ , we have*

$$(2.22) \quad b_{ij}^e(x) \geq b_{ij}^{e,s}(x)$$

*and the linear system*

$$(2.23) \quad -\Delta \psi + B^{e,s}(x)\psi = 0$$

*is fully coupled in  $\Omega$  and in  $\Omega(e)$  as well for any  $e \in S^{N-1}$ , where  $B^{e,s}(x) = (b_{ij}^{e,s}(x))_{i,j=1}^m$ . Finally, if  $U$  is symmetric with respect to the hyperplane  $T(e)$*

*then  $b_{ij}^e(x) = b_{ij}^{e,s}(x) = \frac{\partial f_i}{\partial u_j}(|x|, U(x))$  for any  $i, j = 1, \dots, m$ .*

The proof of this lemma is the same as in the case of bounded domains, see Lemma 3.1 of [2] for case i) and Lemma 3.1 of [3] for case ii). Throughout the paper we will denote by

$$(2.24) \quad Q_{es}(\psi, D) := \int_D \sum_{i=1}^m |\nabla \psi_i|^2 + \sum_{i,j=1}^m b_{ij}^{e,s}(x) \psi_j \psi_i \, dx$$

the quadratic form associated with the linear system (2.23), and by

$$(2.25) \quad P_{es}(\psi, \phi, D) := \int_D \sum_{i=1}^m \nabla \psi_i \cdot \nabla \phi_i + \frac{1}{2} \sum_{i,j=1}^m (b_{ij}^{e,s}(x) + b_{ji}^{e,s}(x)) \psi_j \phi_i \, dx$$

the corresponding bilinear symmetric form.

Now we recall the following result.

**Lemma 2.5.** *Assume that  $|\nabla U| \in L^2(\Omega)$ . Then, for any  $e \in S^{N-1}$  and for any  $j = 1, \dots, m$ , we have*

$$(2.26) \quad \frac{1}{R^2} \int_{B_{2R} \setminus B_R} w_j^2 dx \rightarrow 0 \quad \text{as } R \rightarrow \infty$$

where  $w_j$  are the components of the function  $W^e$ .

See Lemma 2.2 in [9] for a detailed proof.

**Lemma 2.6.** *Let  $U$  be a solution of (1.1) and (1.2), such that the system (1.1) is fully coupled along  $U$  in  $\Omega(e)$  for any  $e \in S^{N-1}$ . Then there exists  $\bar{R} > 0$  such that the system (1.1) is fully coupled along  $U$  in  $\Omega(e) \cap B_R$  for any  $R > \bar{R}$  and for any  $e \in S^{N-1}$ .*

*Proof.* Assume, by contradiction, that there exists a sequence of radii  $R_n \rightarrow +\infty$ , a sequence of directions  $e_n \in S^{N-1}$  and a sequence of subsets  $I_n \subset \{1, \dots, m\}$  such that

$$(2.27) \quad \text{meas} \left( \left\{ x \in \Omega(e_n) \cap B_{R_n}, \frac{\partial f_{i_n}}{\partial u_{j_n}}(|x|, U) > 0 \right\} \right) = 0$$

for any  $i_n \in I_n$  and for any  $j_n \in \{1, \dots, m\} \setminus I_n$ .

Since  $I_n \subset \{1, \dots, m\}$  there exists  $I \subset \{1, \dots, m\}$ ,  $I \neq \emptyset$  and a subsequence  $s(n)$  such that  $I_{s(n)} = I$  for any  $n \in \mathbb{N}$ . Up to a subsequence  $e_{s(n)} \rightarrow e \in S^{N-1}$  and  $\Omega(e_{s(n)}) \cap B_{R_{s(n)}} \rightarrow \Omega(e)$ .

The hypothesis of the fully coupling in  $\Omega(e)$  implies that there exist  $i_0 \in I$  and  $j_0 \in \{1, \dots, m\} \setminus I$  such that

$$\text{meas} \left( \left\{ x \in \Omega(e), \frac{\partial f_{i_0}}{\partial u_{j_0}}(|x|, U) > 0 \right\} \right) > 0.$$

Then there exist  $x \in \Omega(e)$  and  $B_\rho(x) \subset \Omega(e)$  such that  $B_\rho(x) \subset \left\{ x \in \Omega(e), \frac{\partial f_{i_0}}{\partial u_{j_0}}(|x|, U) > 0 \right\}$ . By continuity  $B_\rho(x) \subset \Omega(e_{s(n)}) \cap B_{R_{s(n)}}$  contradicting (2.27).  $\square$

**Remark 2.7.** The same proof of the previous Lemma shows also that, if  $U$  is a solution of (1.1) and (1.2), such that the system (1.1) is fully coupled along  $U$  in  $\Omega$  then there exists  $\bar{R} > 0$  such that the system (1.1) is fully coupled along  $U$  in  $\Omega \cap B_R$  for any  $R > \bar{R}$ . Moreover by *ii*) of Lemma 2.4 we also have that the linear system (2.23) is fully coupled in  $\Omega(e) \cap B_R$  for any  $R > \bar{R}$  and for any  $e \in S^{N-1}$ .

Let  $U$  be a solution of (1.1) and (1.2) and let  $D \subseteq \Omega$ . We will denote by  $L_U$  the linearized operator at  $U$ , i.e.  $L_U = -\Delta - J_F(|x|, U)$  where  $J_F(|x|, U)$  is the jacobian matrix of  $F$  computed at  $U$ , and we define the linearized system at  $U$ , i.e.

$$(2.28) \quad \begin{cases} -\Delta \psi - J_F(|x|, U) \psi & \text{in } D \\ \psi = 0 & \text{on } \partial D. \end{cases}$$

Associated with system (1.1), or with system (2.28), we can consider the quadratic form

$$(2.29) \quad Q_U(\psi, D) := \int_D \sum_{i=1}^m |\nabla \psi_i|^2 - \sum_{i,j=1}^m \frac{\partial f_i}{\partial u_j}(|x|, U) \psi_j \psi_i dx$$

and the bilinear symmetric form

(2.30)

$$P_U(\psi, \phi, D) := \int_D \sum_{i=1}^m \nabla \psi_i \cdot \nabla \phi_i - \frac{1}{2} \sum_{i,j=1}^m \left( \frac{\partial f_i}{\partial u_j}(|x|, U) + \frac{\partial f_j}{\partial u_i}(|x|, U) \right) \psi_j \phi_i dx$$

for any  $\psi, \phi \in C_c^1(D, \mathbb{R}^m)$  (but also for  $\psi, \phi \in H_0^1(D, \mathbb{R}^m)$  which vanish outside a bounded set). In the sequel we will denote by  $J_F^t(|x|, U)$  the transpose matrix of  $J_F(|x|, U)$ .

For an arbitrary subset  $D$  of  $\mathbb{R}^N$  we let  $\chi_D : \mathbb{R}^N \rightarrow \mathbb{R}$  denote the characteristic function of  $D$ . To avoid complicated notation, we denote the restriction of  $\chi_D$  to arbitrary subsets of  $\mathbb{R}^N$  again by  $\chi_D$ .

We introduce a family of cutoff functions which we will be frequently used throughout the paper. To this aim we fix a  $C^\infty$ -function  $\xi$  defined on  $[0, \infty)$  such that  $0 \leq \xi \leq 1$  and

$$\xi(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ 0 & \text{if } t \geq 2. \end{cases}$$

For  $R > 0$ , we then define

$$(2.31) \quad \xi_R \in C_c^\infty(\mathbb{R}^N, \mathbb{R}), \quad \xi_R(x) = \xi\left(\frac{|x|}{R}\right).$$

We will denote the restriction of  $\xi_R$  to arbitrary subsets of  $\mathbb{R}^N$  again by  $\xi_R$ .

**Lemma 2.8.** *Let  $U$  be a solution of (1.1) and (1.2) such that  $|\nabla U| \in L^2(\Omega)$ . Let  $e \in S^{N-1}$  and  $D \subset \Omega(e)$  be a (possibly unbounded) open set such that  $W_{|\partial D}^e \equiv 0$ . Then*

- i) *if assumptions i), ii) and iii) of Theorem 1.1 are satisfied and if  $W^e \geq 0$  in  $D$ , we have*

$$(2.32) \quad \limsup_{R \rightarrow +\infty} Q_U(W^e \chi_D \xi_R, \Omega(e)) \leq 0$$

*where  $Q_U$  is as in (2.29).*

- ii) *if assumptions i), and ii) of Theorem 1.2 are satisfied, we have*

$$(2.33) \quad \limsup_{R \rightarrow +\infty} Q_{es}((W^e)^\pm \chi_D \xi_R, \Omega(e)) \leq 0$$

*where  $Q_{es}(-, D)$  is as defined in (2.24).*

Note that since  $W_{|\partial D}^e \equiv 0$ , the function  $W^e \chi_D \xi_R$  belongs to  $H_0^1(\Omega(e), \mathbb{R}^m)$  for  $R > 0$  and vanishes a.e. outside a bounded subset of  $\Omega(e)$ . Hence the quadratic forms  $Q_U$  and  $Q_{es}$  are well defined on these functions.

*Proof.* i) The function  $W^e = (w_1, \dots, w_m)$  satisfies the linear system (2.18) in  $\Omega(e)$ . Multiplying the  $i$ -th equation of the linear system (2.18) by  $w_i \chi_D \xi_R^2$  and integrating

on  $\Omega(e)$  we get

$$\begin{aligned} 0 &= \int_{\Omega(e)} (-\Delta w_i)(w_i \chi_D \xi_R^2) dx + \sum_{j=1}^m \int_{\Omega(e)} \tilde{b}_{ij}^e(x) w_j (w_i \chi_D \xi_R^2) dx \\ &= \int_D \nabla w_i \cdot \nabla (w_i \xi_R^2) dx + \sum_{j=1}^m \int_D \tilde{b}_{ij}^e(x) w_j (w_i \xi_R^2) dx. \end{aligned}$$

Using (2.19) and (2.20), we deduce

$$0 \geq \int_D \nabla w_i \cdot \nabla (w_i \xi_R^2) dx - \sum_{j=1}^m \int_D \frac{\partial f_i}{\partial u_j}(|x|, U)(w_j \xi_R)(w_i \xi_R) dx$$

for any  $i = 1, \dots, m$ . Letting  $v_R := W^e \chi_D \xi_R$ , we have, from the definition of  $Q_U$ , that

$$\begin{aligned} Q_U(v_R, \Omega(e)) &= \sum_{i=1}^m \int_{\Omega(e)} |\nabla w_i \chi_D \xi_R|^2 dx - \sum_{i,j=1}^m \int_{\Omega(e)} \frac{\partial f_i}{\partial u_j}(|x|, U)(w_j \chi_D \xi_R)(w_i \chi_D \xi_R) dx \\ &= \sum_{i=1}^m \int_D [w_i^2 |\nabla \xi_R|^2 + \nabla w_i \cdot \nabla (w_i \xi_R^2)] dx - \sum_{i,j=1}^m \int_D \frac{\partial f_i}{\partial u_j}(|x|, U)(w_j \xi_R)(w_i \xi_R) dx \\ &= \sum_{i=1}^m \int_D \nabla w_i \cdot \nabla (w_i \xi_R^2) - \sum_{j=1}^m \frac{\partial f_i}{\partial u_j}(|x|, U)(w_j \xi_R)(w_i \xi_R) dx + \sum_{i=1}^m \int_D w_i^2 |\nabla \xi_R|^2 dx \\ &\leq \sum_{i=1}^m \int_D w_i^2 |\nabla \xi_R|^2 dx \leq \sum_{i=1}^m \frac{C}{R^2} \int_{B_{2R} \setminus B_R} w_i^2 dx \rightarrow 0, \end{aligned}$$

having applied Lemma 2.5. Thus i) is proved.

ii) We give the proof of (2.33) for  $(W^e)^+$ . The case of  $(W^e)^-$  follows in the same way. The function  $W^e$  satisfies the linear system (2.15) in  $\Omega(e)$ . Multiplying the  $i$ -th equation of this system by  $w_i^+ \chi_D \xi_R^2$  and integrating on  $\Omega(e)$  we get

$$\int_{\Omega(e)} (\Delta w_i) (w_i^+ \chi_D \xi_R^2) dx = - \int_D \nabla w_i \cdot \nabla (w_i^+ \xi_R^2) dx = \sum_{j=1}^m \int_D b_{ij}^e(x) w_j w_i^+ \xi_R^2 dx$$

for  $i = 1, \dots, m$ , so that, letting  $v_R := (W^e)^+ \chi_D \xi_R$ , we have

$$\int_{\Omega(e)} |\nabla v_R|^2 dx = \sum_{i=1}^m \int_D (w_i^+)^2 |\nabla \xi_R|^2 dx - \sum_{i,j=1}^m \int_D b_{ij}^e(x) w_j w_i^+ \xi_R^2 dx.$$

Then, the definition of  $Q_{es}$  implies that

$$\begin{aligned}
Q_{es}(v_R, \Omega(e)) &= \int_{\Omega(e)} |\nabla v_R|^2 dx + \sum_{i,j=1}^m \int_{\Omega(e)} b_{ij}^{es}(x) w_j^+ w_i^+ \chi_D \xi_R^2 dx \\
&= \sum_{i=1}^m \int_D (w_i^+)^2 |\nabla \xi_R|^2 dx - \sum_{i,j=1}^m \int_D b_{ij}^e(x) (w_j^+ - w_j^-) w_i^+ \xi_R^2 dx \\
&\quad + \sum_{i,j=1}^m \int_D b_{ij}^{es}(x) w_j^+ w_i^+ \xi_R^2 dx \\
&= \sum_{i,j=1}^m \int_D (b_{ij}^{es}(x) - b_{ij}^e(x)) w_j^+ w_i^+ \xi_R^2 dx \\
&\quad + \sum_{\substack{i,j=1 \\ i \neq j}}^m \int_D b_{ij}^e(x) w_j^- w_i^+ \xi_R^2 dx + \sum_{i=1}^m \int_D (w_i^+)^2 |\nabla \xi_R|^2 dx.
\end{aligned}$$

Thus from Lemma 2.4 we get that  $b_{ij}^e(x) \leq 0$  for  $i \neq j$  and that  $b_{ij}^{es}(x) \leq b_{ij}^e(x)$ . Then

$$Q_{es}(v_R, \Omega(e)) \leq \limsup_{R \rightarrow +\infty} \frac{C}{R^2} \sum_{i=1}^m \int_{B_{2R} \setminus B_R} (w_i^+)^2 dx$$

and (2.33) follows again by Lemma 2.5.  $\square$

### 3. Sufficient conditions for foliated Schwarz symmetry

**Lemma 3.1.** *Let  $U = (u_1, \dots, u_m)$  be a solution of (1.1) and (1.2) and assume that the hypothesis i) either of Theorem 1.1 or of Theorem 1.2 holds. If for every  $e \in S^{N-1}$  we have either  $U \geq U^{\sigma_e}$  in  $\Omega(e)$  or  $U \leq U^{\sigma_e}$  in  $\Omega(e)$ , then  $U$  is foliated Schwarz symmetric.*

The proof is exactly the same as in the case of bounded domains, see Lemma 3.2 in [2] and Lemma 3.2 in [3] as well as [16] for the scalar case.

We will now describe other sufficient conditions for the foliated Schwarz symmetry of a solution  $U$  of (1.1) and (1.2).

To this end we begin with some geometric considerations about cylindrical coordinates with respect to the plane  $x_1 x_2$ .

Suppose  $\beta \in \mathbb{R}$  and let  $e_\beta = (\cos \beta, \sin \beta, 0, \dots, 0)$  be a unit vector in the  $x_1 x_2$  plane. Then we consider as before the hyperplane  $T(e_\beta)$  and, for simplicity, we will use the notations  $\Omega_\beta = \Omega(e_\beta)$ ,  $T_\beta = T(e_\beta)$ ,  $\sigma_\beta = \sigma_{e_\beta}$ .

Using cylindrical coordinates we will write  $x = (x_1, \dots, x_N)$  as  $x = (r, \theta, \tilde{x}) = (r \cos \theta, r \sin \theta, \tilde{x})$  where  $r = \sqrt{x_1^2 + x_2^2}$ ,  $\tilde{x} = (x_3, \dots, x_N)$  and  $\theta \in [0, 2\pi]$ .

It is easy to see that the reflection  $\sigma_\beta$  through  $T_\beta$  can be written as

$$(3.34) \quad \sigma_\beta(r \cos \theta, r \sin \theta, \tilde{x}) = (r \cos(2\beta - \theta + \pi), r \sin(2\beta - \theta + \pi), \tilde{x})$$

(in fact  $2\beta - \theta + \pi = \theta + 2(\beta + \frac{\pi}{2} - \theta)$ ). It can also be proved analytically writing the usual reflection in cartesian coordinates and using simple trigonometry formulas.

Of course, since the angular variable is defined up to a multiple of  $2\pi$ , we could also write the angular variable of the point  $\sigma_\beta(r \cos \theta, r \sin \theta, \tilde{x})$  as  $2\beta - \theta - \pi$ .

Let us put  $h_\beta^\pm(\theta) = 2\beta - \theta \pm \pi$ .

Note that if we choose an interval  $[\theta_0, \theta_0 + 2\pi)$  to which the angular coordinate  $\theta$  belongs, the images  $2\beta - \theta \pm \pi$  could not belong to the same interval. Nevertheless we observe that for a fixed  $\tilde{\beta} \in \mathbb{R}$ , if we take  $\theta \in [\tilde{\beta} - \frac{\pi}{2}, \tilde{\beta} + \frac{3}{2}\pi]$  then  $h_{\tilde{\beta}}^+(\theta)$  belongs to the same interval, whereas if we take  $\theta \in [\tilde{\beta} - \frac{3}{2}\pi, \tilde{\beta} + \frac{\pi}{2}]$  then  $h_{\tilde{\beta}}^-(\theta)$  belongs to the same interval. More precisely we have

$$(3.35) \quad \theta \in [\tilde{\beta} - \frac{\pi}{2}, \tilde{\beta} + \frac{\pi}{2}] \Rightarrow 2\tilde{\beta} - \theta + \pi = h_{\tilde{\beta}}^+(\theta) \in [\tilde{\beta} + \frac{\pi}{2}, \tilde{\beta} + \frac{3}{2}\pi],$$

$$(3.36) \quad \theta \in [\tilde{\beta} - \frac{3}{2}\pi, \tilde{\beta} - \frac{\pi}{2}] \Rightarrow 2\tilde{\beta} - \theta - \pi = h_{\tilde{\beta}}^-(\theta) \in [\tilde{\beta} - \frac{\pi}{2}, \tilde{\beta} + \frac{\pi}{2}].$$

This can be easily verified evaluating  $h_{\tilde{\beta}}^\pm$  in the boundary of the intervals of definition, since the mappings  $h_{\tilde{\beta}}^\pm$  are decreasing.

Let us denote by  $U_\theta(r, \theta, \tilde{x})$  the derivative of the function  $U$  with respect to the angular coordinate  $\theta$ .

**Proposition 3.2.** *Let  $\tilde{\beta} \in \mathbb{R}$  and assume that  $U$  is symmetric with respect to the hyperplane  $T_{\tilde{\beta}}$ . If  $U_\theta \geq 0$  in  $\Omega_{\tilde{\beta}} = [x \cdot e_{\tilde{\beta}} > 0]$ , then for any  $\beta \in [\tilde{\beta} - \pi, \tilde{\beta}]$  and for any  $x \in \Omega_\beta = [x \cdot e_\beta > 0]$  we have  $U(x) \leq U(\sigma_\beta(x))$ , while for every  $\beta \in [\tilde{\beta}, \tilde{\beta} + \pi]$  we have  $U \geq U^{\sigma_\beta}$  in  $\Omega_\beta$ .*

To prove this proposition let us first state and prove a simple lemma.

**Lemma 3.3.** *Suppose that  $t_0 \in (\tilde{\beta} - \frac{\pi}{2}, \tilde{\beta} + \frac{\pi}{2}]$  and that the assumptions of Proposition 3.2 hold. Then*

$$(3.37) \quad U(r, t, \tilde{x}) \geq U(r, t_0, \tilde{x}) \quad \forall t \in [t_0, 2\tilde{\beta} - t_0 + \pi].$$

*Proof.* Since  $U$  is symmetric,  $U_\theta$  is antisymmetric with respect to  $T_{\tilde{\beta}}$ . By hypothesis  $U_\theta \geq 0$  in  $\Omega_{\tilde{\beta}} = [x \cdot e_{\tilde{\beta}} > 0] = \{(r, \theta, \tilde{x}) : \tilde{\beta} - \frac{\pi}{2} \leq \theta \leq \tilde{\beta} + \frac{\pi}{2}\}$  so that  $U_\theta(r, \theta, \tilde{x}) \leq 0$  if  $\tilde{\beta} + \frac{\pi}{2} \leq \theta \leq \tilde{\beta} + \frac{3}{2}\pi$ . Moreover by (3.35), if  $\tilde{\beta} - \frac{\pi}{2} < t_0 < \tilde{\beta} + \frac{\pi}{2}$  then  $h_{\tilde{\beta}}^+(t_0) = 2\tilde{\beta} - t_0 + \pi \in [\tilde{\beta} + \frac{\pi}{2}, \tilde{\beta} + \frac{3}{2}\pi]$ . This means that  $U(r, \cdot, \tilde{x})$  increases in  $[t_0, \tilde{\beta} + \frac{\pi}{2}]$ , then decreases in  $[\tilde{\beta} + \frac{\pi}{2}, \tilde{\beta} + \frac{3}{2}\pi]$ , in particular in  $[\tilde{\beta} + \frac{\pi}{2}, 2\tilde{\beta} - t_0 + \pi]$ . Since  $U(r, t_0, \tilde{x}) = U(r, 2\tilde{\beta} - t_0 + \pi, \tilde{x})$ , (3.37) follows.  $\square$

*Proof of Proposition 3.2.* Let  $\beta \in [\tilde{\beta} - \pi, \tilde{\beta}]$  and let  $x \in \Omega_\beta = [x \cdot e_\beta > 0]$ , equivalently  $x = (r \cos \theta, r \sin \theta, \tilde{x})$  with  $\beta - \frac{\pi}{2} < \theta < \beta + \frac{\pi}{2}$ . We have to show that  $U(x) \leq U^{\sigma_\beta}(x)$  if  $\beta - \frac{\pi}{2} < \theta < \beta + \frac{\pi}{2}$ ,  $\tilde{\beta} - \pi \leq \beta \leq \tilde{\beta}$ .

Let us observe that, since  $U$  is symmetric with respect to  $T_{\tilde{\beta}}$ ,

$$U^{\sigma_\beta}(x) = U(\sigma_{\tilde{\beta}}(\sigma_\beta(x))) = U(r \cos(\theta + 2(\tilde{\beta} - \beta)), r \sin(\theta + 2(\tilde{\beta} - \beta)), \tilde{x})$$

because  $2\tilde{\beta} - (2\beta - \theta + \pi) + \pi = \theta + 2(\tilde{\beta} - \beta)$ .

Let us first assume that  $x \in \Omega_\beta \cap \Omega_{\tilde{\beta}}$ , i.e.  $x = (r \cos \theta, r \sin \theta, \tilde{x})$  with  $(\beta - \frac{\pi}{2} \leq) \tilde{\beta} - \frac{\pi}{2} < \theta < \beta + \frac{\pi}{2} (\leq \tilde{\beta} + \frac{\pi}{2})$ . Then we can apply Lemma 3.3 taking  $t_0 = \theta$ ,  $t = \theta + 2(\tilde{\beta} - \beta)$ : we have that  $\tilde{\beta} - \frac{\pi}{2} < t_0 = \theta \leq \theta + 2(\tilde{\beta} - \beta) \leq 2\tilde{\beta} - \theta + \pi$ . In fact  $\tilde{\beta} - \beta \geq 0$ , and the last equality is equivalent to  $\theta < \beta + \frac{\pi}{2}$ , which is true, since  $x \in \Omega_\beta$ . So from Lemma 3.3 it follows that

$$U^{\sigma_\beta}(x) = U(r, \theta + 2(\tilde{\beta} - \beta), \tilde{x}) \geq U(r, \theta, \tilde{x}) = U(x).$$

If instead  $x = (r, \theta, \tilde{x}) \in \Omega_\beta \setminus \Omega_{\tilde{\beta}}$ , i.e.  $(\tilde{\beta} - \frac{3}{2}\pi \leq) \beta - \frac{\pi}{2} < \theta \leq \tilde{\beta} - \frac{\pi}{2} (\leq \beta + \frac{\pi}{2})$ , then by (3.36)  $t_0 := 2\tilde{\beta} - \theta - \pi \in [\tilde{\beta} - \frac{\pi}{2}, \tilde{\beta} + \frac{\pi}{2}]$  and  $U(x) = U(r, \theta, \tilde{x}) = U(r, t_0, \tilde{x})$ , while  $U^{\sigma_\beta}(x) = U(r, \theta + 2(\tilde{\beta} - \beta), \tilde{x})$  so that, as before the inequality follows if we show that  $t_0 = 2\tilde{\beta} - \theta - \pi \leq \theta + 2(\tilde{\beta} - \beta) \leq 2\tilde{\beta} - t_0 + \pi = \theta + 2\pi$ . The inequality  $\theta + 2(\tilde{\beta} - \beta) \leq \theta + 2\pi$  follows from the relation  $\tilde{\beta} - \pi \leq \beta \leq \tilde{\beta}$ , while  $2\tilde{\beta} - \theta - \pi \leq \theta + 2(\tilde{\beta} - \beta)$  is equivalent to  $\theta > \beta - \frac{\pi}{2}$ , which is true since  $x \in \Omega_\beta$ . Then by Lemma 3.1 also the foliated Schwarz symmetry follows.  $\square$

**Remark 3.4.** From Proposition 3.2 it follows that if  $\tilde{e}$  is a symmetry direction for  $U$  and for any other direction  $e' \neq \tilde{e}$  in the plane  $\pi(\tilde{e}, e')$  generated by  $\tilde{e}$  and  $e'$ , using the cylindrical coordinates with respect to  $\pi$ , one has  $U_\theta \geq 0$  in  $\Omega(\tilde{e})$ , then  $U$  is foliated Schwarz symmetric, as a consequence of Lemma 3.1. This is the strategy of Proposition 3.7 exploiting condition (3.38).

**Proposition 3.5.** *Let  $\tilde{\beta} \in \mathbb{R}$  and assume that  $U$  is symmetric whit respect to  $T_{\tilde{\beta}}$ . Assume further that there exists  $\beta_1 < \tilde{\beta}$  such that for any  $\beta \in (\beta_1, \tilde{\beta})$  we have  $U \leq U^{\sigma_\beta}$  in  $\Omega_\beta$ . Then  $U_\theta = \frac{\partial U}{\partial \theta} \geq 0$  in  $\Omega_{\tilde{\beta}}$ .*

*Proof.* We can write the angular derivative as

$$U_\theta(r, \theta, \tilde{x}) = \lim_{\alpha \rightarrow 0^+} \frac{U(r, \theta + \alpha, \tilde{x}) - U(r, \theta, \tilde{x})}{\alpha}.$$

With the change of variable  $\alpha = 2(\tilde{\beta} - \beta)$ ,  $\beta = \tilde{\beta} - \frac{\alpha}{2}$ , we have that  $\beta \rightarrow \tilde{\beta}^-$ . If  $\alpha$  is small then  $\beta \in (\beta_1, \tilde{\beta})$ , and, if  $x \in \Omega_{\tilde{\beta}}$ , then  $x \in \Omega_\beta$  definitively for  $\beta \rightarrow \tilde{\beta}^-$ . Since  $U^{\sigma_\beta}(r, \theta, \tilde{x}) = U(r, \theta + 2(\tilde{\beta} - \beta), \tilde{x})$  as observed, we obtain that

$$\begin{aligned} U_\theta(r, \theta, \tilde{x}) &= \lim_{\beta \rightarrow \tilde{\beta}^-} \frac{U(r, \theta + 2(\tilde{\beta} - \beta), \tilde{x}) - U(r, \theta, \tilde{x})}{2(\tilde{\beta} - \beta)} \\ &= \lim_{\beta \rightarrow \tilde{\beta}^-} \frac{U^{\sigma_\beta}(r, \theta, \tilde{x}) - U(r, \theta, \tilde{x})}{2(\tilde{\beta} - \beta)} \geq 0. \end{aligned}$$

$\square$

**Remark 3.6.** Let us remark that if there exists a direction  $\bar{e} \in \mathcal{S}^{N-1}$  such that  $W^{\bar{e}}(x) > 0$ , for any  $x \in \Omega(\bar{e})$  and if a rotating plane argument can be applied to the solution  $U$ , then the foliated Schwarz symmetry of  $U$  follows from the previous propositions. Indeed, if  $e'$  is any other direction,  $e' \neq \bar{e}$ , rotating the hyperplane  $T(\bar{e})$  we get a symmetry hyperplane  $T(\tilde{e})$ , with  $\tilde{e}$  belonging to the 2-dimensional plane generated by  $e'$  and  $\bar{e}$ . (see the proof of Proposition 3.9 below for details). Then Proposition 3.5 applies and by Lemma 3.1 we obtain the foliated Schwarz symmetry of  $U$ .

Let us further notice that in the case of a single equation ( $m = 1$ ) or of ssystem but in bounded domains the rotating plane method can be applied without further assumptions, so the previous remark yelds the foliated Schwarz symmetry of  $U$ . In the scalar case, for bounded domains this was also observed in [14], (Corollary 1.2) by a different kind of argument.

In the case of systems in unbounded domains we need to work under the hypothesis of Theorem 1.1 or Theorem 1.2 to perform the rotating plane method. We will use this procedure to prove the foliated Schwarz symmetry of the solution of (1.1) and

(1.2) in Proposition 3.9.

We now give another sufficient condition for foliated Schwarz symmetry.

**Proposition 3.7.** *Let  $U$  be a solution of (1.1) and (1.2) such that  $|\nabla U| \in L^2(\Omega)$ , and assume that the system (1.1) is fully coupled along  $U$  in  $\Omega$ . Suppose further that there exists  $e \in S^{N-1}$  such that  $U$  is symmetric with respect to the hyperplane  $T(e)$  and*

$$(3.38) \quad \inf_{\psi \in C_c^1(\Omega(e), \mathbb{R}^m)} Q_U(\psi, \Omega(e)) \geq 0.$$

*Then  $U$  is foliated Schwarz symmetric.*

*Proof.* First observe that the symmetry of  $U$  with respect to  $T(e)$  and the coupling conditions imply that the system (1.1) is fully coupled along  $U$  also in  $\Omega(e)$ .

We want to prove the foliated Schwarz symmetry of  $U$  using Proposition 3.2 and Lemma 3.1.

We follow the proof of Proposition 2.5 in [9]. We may assume that  $e = e_2 = (0, 1, \dots, 0)$ , so that  $T(e) = \{x \in \mathbb{R}^N : x_2 = 0\}$ . We consider an arbitrary unit vector  $e' \in S^{N-1}$  different from  $\pm e$ . After another orthogonal transformation which leaves  $e_2$  and  $T(e_2)$  invariant, we may assume that  $e' = e_\beta = (\cos \beta, \sin \beta, 0, \dots, 0)$  for some  $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ . Now we consider the cylindrical coordinates  $(r, \theta, \tilde{x})$  introduced before. The derivative of  $U$  with respect to  $\theta$  - denoted by  $U_\theta$  - extended in the origin with  $U_\theta(0) = 0$  if  $\Omega = \mathbb{R}^N$ , satisfies the linearized system

$$(3.39) \quad \begin{cases} -\Delta U_\theta^1 = \sum_{j=1}^m \frac{\partial f_1}{\partial u_j}(|x|, U) U_\theta^j & \text{in } \Omega(e) \\ \dots\dots\dots \\ -\Delta U_\theta^m = \sum_{j=1}^m \frac{\partial f_m}{\partial u_j}(|x|, U) U_\theta^j & \text{in } \Omega(e) \\ U_\theta^1 = \dots = U_\theta^m = 0 & \text{on } \partial\Omega(e). \end{cases}$$

All we need is to show that  $U_\theta$  does not change sign in  $\Omega(e)$ . Indeed, if this is the case, Proposition 3.2 implies that either  $U \leq U_{\sigma_{e_\beta}}$  or  $U \geq U_{\sigma_{e_\beta}}$  in  $\Omega(e_\beta)$  and by the arbitrariness of  $e_\beta$  Lemma 3.1 implies the foliated Schwarz symmetry of  $U$ .

We first note that, by (3.38), the bilinear form  $P_U(\psi, \phi, \Omega(e))$ , defined in (2.30), defines a (semidefinite) scalar product on  $C_c^1(\Omega(e), \mathbb{R}^m) \times C_c^1(\Omega(e), \mathbb{R}^m)$ , and the corresponding Cauchy-Schwarz-inequality yields:

$$(3.40) \quad (P_U(\psi, \phi, \Omega(e)))^2 \leq P_U(\psi, \psi, \Omega(e)) P_U(\phi, \phi, \Omega(e)) = Q_U(\psi, \Omega(e)) Q_U(\phi, \Omega(e))$$

for any  $\psi, \phi$  are  $H_0^1(\Omega(e), \mathbb{R}^m)$ -functions vanishing a.e. outside a bounded set. We consider  $\xi_R$  as defined in (2.31) and, for  $R > 0$ , we let

$$v_R \in H_0^1(\Omega(e), \mathbb{R}^m), \quad v_R(x) = \xi_R(x) U_\theta^+(x) \quad \text{for any } x \in \Omega(e).$$

We also fix  $\phi \in C_c^1(\Omega(e), \mathbb{R}^m)$ . Then (3.40) yields

$$(3.41) \quad \begin{aligned} & \left( \int_{\Omega(e)} \nabla U_\theta^+ \cdot \nabla \phi - \frac{1}{2} \sum_{i,j=1}^m \left( \frac{\partial f_i}{\partial u_j}(|x|, U) + \frac{\partial f_j}{\partial u_i}(|x|, U) \right) (U_\theta^j)^+ \phi^i dx \right)^2 \\ &= \lim_{R \rightarrow +\infty} (P_U(v_R, \phi, \Omega(e)))^2 \leq Q_U(\phi, \Omega(e)) \limsup_{R \rightarrow +\infty} Q_U(v_R, \Omega(e)). \end{aligned}$$



Moreover, since

$$\begin{aligned} \int_{\Omega(e)} |\nabla v_R|^2 dx &= \sum_{i=1}^m \int_{\Omega(e)} |\nabla (\xi_R (U_\theta^i)^+)|^2 dx \\ &= \sum_{i=1}^m \int_{\Omega(e)} ((U_\theta^i)^+)^2 |\nabla \xi_R|^2 dx + \sum_{i=1}^m \int_{\Omega(e)} \nabla (U_\theta^i)^+ \cdot \nabla (\xi_R^2 (U_\theta^i)^+) dx \end{aligned}$$

from (3.39) we have

$$\begin{aligned} \int_{\Omega(e)} |\nabla v_R|^2 dx - \int_{\Omega(e)} |U_\theta^+|^2 |\nabla \xi_R|^2 dx &= \int_{\Omega(e)} \nabla U_\theta^+ \cdot \nabla (\xi_R^2 U_\theta^+) dx \\ &= \sum_{i=1}^m \int_{\Omega(e) \cap \{U_\theta^i > 0\}} \nabla U_\theta^i \cdot \nabla (\xi_R^2 (U_\theta^i)^+) dx \\ &= \sum_{i=1}^m \int_{\Omega(e) \cap \{U_\theta^i > 0\}} [-\Delta U_\theta^i] (\xi_R^2 (U_\theta^i)^+) dx \\ &= \sum_{i=1}^m \int_{\Omega(e) \cap \{U_\theta^i > 0\}} \sum_{j=1}^m \frac{\partial f_i}{\partial u_j}(|x|, U) U_\theta^j (U_\theta^i)^+ \xi_R^2 dx \\ &= \sum_{i,j=1}^m \int_{\Omega(e) \cap \{U_\theta^i > 0\}} \frac{\partial f_i}{\partial u_j}(|x|, U) (\xi_R U_\theta^j) (\xi_R (U_\theta^i)^+) dx \\ &= \sum_{i,j=1}^m \int_{\Omega(e) \cap \{U_\theta^i > 0\}} \frac{\partial f_i}{\partial u_j}(|x|, U) (\xi_R ((U_\theta^j)^+ - (U_\theta^j)^-)) v_R^i dx \\ &= \sum_{i,j=1}^m \int_{\Omega(e)} \frac{\partial f_i}{\partial u_j}(|x|, U) v_R^j v_R^i dx - \sum_{\substack{i,j=1 \\ i \neq j}}^m \int_{\Omega(e)} \frac{\partial f_i}{\partial u_j}(|x|, U) \xi_R (U_\theta^j)^- v_R^i dx \end{aligned}$$

and since  $\frac{\partial f_i}{\partial u_j}(|x|, U) \geq 0$  for  $i \neq j$ , and  $\xi_R (U_\theta^j)^- v_R^i \geq 0$  then

$$\int_{\Omega(e)} |\nabla v_R|^2 dx - \int_{\Omega(e)} |U_\theta^+|^2 |\nabla \xi_R|^2 dx \leq \sum_{i,j=1}^m \int_{\Omega(e)} \frac{\partial f_i}{\partial u_j}(|x|, U) v_R^j v_R^i dx.$$

Then

$$\begin{aligned} Q_U(v_R, \Omega(e)) &= \int_{\Omega(e)} (|\nabla v_R|^2 - \sum_{i,j=1}^m \frac{\partial f_i}{\partial u_j}(|x|, U) v_R^j v_R^i) dx \\ &\leq \int_{\Omega(e)} |U_\theta^+|^2 |\nabla \xi_R|^2 dx \leq \frac{1}{R^2} \int_{B_{2R} \setminus B_R} |U_\theta|^2 dx \\ &\leq \frac{1}{R^2} \int_{B_{2R} \setminus B_R} |x|^2 |\nabla U|^2 dx \leq 4 \int_{B_{2R} \setminus B_R} |\nabla U|^2 dx. \end{aligned}$$

Since  $|\nabla U| \in L^2(\Omega)$  we conclude that  $\limsup_{R \rightarrow \infty} Q_U(v_R, \Omega(e)) \leq 0$ , so that from (3.41) we have

$$\int_{\Omega(e)} \nabla U_\theta^+ \cdot \nabla \phi - \frac{1}{2} \sum_{i,j=1}^m \left( \frac{\partial f_i}{\partial u_j}(|x|, U) + \frac{\partial f_j}{\partial u_i}(|x|, U) \right) (U_\theta^j)^+ \phi^i dx = 0.$$

Since  $\phi \in C_c^1(\Omega(e), \mathbb{R}^m)$  was chosen arbitrarily, we conclude that  $U_\theta^+$  is a solution of the symmetric system associated with the linearized system, i.e.  $U_\theta^+$  is a weak solution of

$$\begin{cases} -\Delta (U_\theta^1)^+ - \frac{1}{2} \sum_{j=1}^m \left( \frac{\partial f_1}{\partial u_j}(|x|, U) + \frac{\partial f_j}{\partial u_1}(|x|, U) \right) (U_\theta^j)^+ = 0 & \text{in } \Omega(e) \\ \dots\dots\dots \\ -\Delta (U_\theta^m)^+ - \frac{1}{2} \sum_{j=1}^m \left( \frac{\partial f_m}{\partial u_j}(|x|, U) + \frac{\partial f_j}{\partial u_m}(|x|, U) \right) (U_\theta^j)^+ = 0 & \text{in } \Omega(e) \\ U_\theta^1 = \dots = U_\theta^m = 0 & \text{on } \partial\Omega(e) \end{cases}$$

and, as remarked before, this symmetric system is fully coupled in  $\Omega(e)$ . The Strong Maximum Principle then implies that either  $U_\theta^+ \equiv 0$  or  $U_\theta^+ > 0$  in  $\Omega(e)$  and this concludes the proof.  $\square$

Now we need to recall the following definition

**Definition 6.** Let  $U$  be a  $C^2(\Omega, \mathbb{R}^m)$  solution of (1.1) and (1.2). We say that  $U$  is stable outside a compact set  $\mathcal{K} \subset \Omega$  if  $Q_U(\psi, \Omega \setminus \mathcal{K}) \geq 0$  for any  $\psi \in C_c^1(\Omega \setminus \mathcal{K}, \mathbb{R}^m)$ .

Obviously, we have

**Remark 3.8.** If  $U$  has finite Morse index, then  $U$  is stable outside a compact set  $\mathcal{K} \subset \Omega$ .

Using the stability outside a compact set, we now derive, by means of a rotating plane argument, the following proposition which guarantees the foliated Schwarz symmetry of  $U$ .

**Proposition 3.9.** Let  $U$  be a solution of (1.1) and (1.2) such that  $|\nabla U| \in L^2(\Omega)$  and that  $U$  is stable outside a compact set  $\mathcal{K} \subset \Omega$ . Assume that one of the following holds:

- \*) suppose that assumptions i), ii) and iii) of Theorem 1.1 are satisfied;
- \*\*) suppose that assumptions i), and ii) of Theorem 1.2 are satisfied.

If there exists a direction  $e \in S^{N-1}$  such that

$$(3.42) \quad W^e(x) > 0 \quad \text{or} \quad W^e(x) < 0 \quad \text{for any } x \in \Omega(e),$$

then  $U$  is foliated Schwarz symmetric.

*Proof.* Let us assume that  $e = (1, 0, \dots, 0)$  and that  $W^e < 0$  in  $\Omega(e)$ . We consider an arbitrary unitary vector  $e' \in S^{N-1}$  different from  $\pm e$ . We want to show that either  $U \geq U^{\sigma_{e'}}$  in  $\Omega(e')$  or  $U \leq U^{\sigma_{e'}}$  in  $\Omega(e')$ . Then, since the vector  $e'$  is arbitrary, the foliated Schwarz symmetry follows from Lemma 3.1.

After an orthogonal change of variable that leaves  $e = e_1$  invariant we can assume  $e' = e_\beta = (\cos \beta, \sin \beta, 0, \dots, 0)$  for some  $\beta \in (0, \pi)$ . We set  $e_\beta = (\cos \beta, \sin \beta, 0, \dots, 0)$  for  $\beta \geq 0$ , so that  $e = e_0$ . As before we write in short

$$\begin{aligned} \Omega_\beta &:= \Omega(e_\beta) = \{x \in \Omega : x_1 \cos \beta + x_2 \sin \beta > 0\} \quad \text{and} \\ W^\beta &:= W^{e_\beta}, \quad T_\beta := T(e_\beta). \end{aligned}$$

Then we start rotating planes and we define

$$\tilde{\beta} = \sup\{\beta \in [0, \pi) : W^{\beta'} \leq 0 \text{ in } \Omega_{\beta'} \text{ for all } \beta' \in [0, \beta)\}.$$

Our aim is to show that  $W^{\tilde{\beta}} \equiv 0$  in  $\Omega_{\tilde{\beta}}$ .

Indeed in this case we can apply Proposition 3.5 getting that the angular derivative

$U_\theta$  in the cylindrical coordinates  $(r, \theta, \tilde{x})$  is nonnegative. Then Proposition 3.2 and Lemma 3.1 give the foliated Schwarz symmetry of  $U$ .

We observe that, by continuity,  $W^{\tilde{\beta}} \leq 0$  and hence  $\tilde{\beta} < \pi$ , because  $W^\pi = -W^0 > 0$  in  $\Omega_\pi = -\Omega_0$ .

Arguing by contradiction we assume that  $W^{\tilde{\beta}} \not\equiv 0$ . The function  $W^{\tilde{\beta}}$  satisfies both the linear systems (2.15) and (2.18) in  $\Omega_{\tilde{\beta}}$ . Moreover if  $*$ ) or  $**$ ) are satisfied then these linear systems are fully coupled in  $\Omega_{\tilde{\beta}}$  by Lemma 2.4. So by the strong maximum principle  $W^{\tilde{\beta}} < 0$  in  $\Omega_{\tilde{\beta}}$ . Moreover, applying the Hopf's Lemma on the hyperplane  $T_{\tilde{\beta}}$ , where  $W^{\tilde{\beta}}$  vanishes, we have, by Theorem 2.1

$$(3.43) \quad \frac{\partial W^{\tilde{\beta}}}{\partial e_{\tilde{\beta}}}(x) < 0 \quad \text{for any } x \in T_{\tilde{\beta}} \cap \Omega.$$

Since, by hypotheses,  $U$  is stable outside a compact set, there exists  $R_0 > 0$  such that

$$(3.44) \quad Q_U(\psi, \Omega \setminus B_{R_0}) \geq 0 \quad \text{for every } \psi \in C_c^1(\Omega \setminus B_{R_0}, \mathbb{R}^m).$$

We fix  $R_1 > R_0$ , and we claim that there exists  $\varepsilon_1 > 0$  such that

$$(3.45) \quad W^{\tilde{\beta}+\varepsilon}(x) \leq 0 \quad \text{in } B_{R_1} \cap \Omega_{\tilde{\beta}+\varepsilon} \quad \forall \varepsilon \in [0, \varepsilon_1].$$

In the case  $\Omega = \mathbb{R}^N \setminus B$ , let  $B_\delta$  be a neighborhood of  $\partial B$  in  $\Omega$  of small measure to allow the maximum principle to hold in  $B_\delta$  for the operator  $-\Delta + \tilde{B}^{\tilde{\beta}+\varepsilon}(x)$  in case  $*$ ) is satisfied or for the operator  $-\Delta + B^{\tilde{\beta}+\varepsilon}(x)$  in case  $**$ ) holds, for sufficiently small  $\varepsilon > 0$ , see Theorem 2.2. We first show that

$$(3.46) \quad W^{\tilde{\beta}+\varepsilon}(x) \leq 0 \quad \text{in } B_{R_1} \cap (\Omega_{\tilde{\beta}+\varepsilon} \setminus B_\delta) \quad \forall \varepsilon \in [0, \varepsilon_1].$$

If (3.46) is not true, we have sequences  $\varepsilon_n \rightarrow 0$  and  $x_n \in B_{R_1} \cap (\Omega_{\tilde{\beta}+\varepsilon_n} \setminus B_\delta)$  such that  $W^{\tilde{\beta}+\varepsilon_n}(x_n) > 0$ . After passing to a subsequence,  $x_n \rightarrow x_0 \in \overline{B_{R_1} \cap (\Omega_{\tilde{\beta}} \setminus B_\delta)}$  and  $W^{\tilde{\beta}}(x_0) = 0$ , hence  $x_0 \in T_{\tilde{\beta}}$ . Since  $W^{\tilde{\beta}+\varepsilon_n}(x) = 0$  on  $T_{\tilde{\beta}+\varepsilon_n}$  and  $W^{\tilde{\beta}+\varepsilon_n}(x_n) > 0$ , there should be points  $\xi_n$  on the line segment joining  $x_n$  with  $T_{\tilde{\beta}+\varepsilon_n}$  and perpendicular to  $T_{\tilde{\beta}+\varepsilon_n}$ , such that  $\frac{\partial W^{\tilde{\beta}+\varepsilon_n}}{\partial e_{\tilde{\beta}+\varepsilon_n}}(\xi_n) > 0$ . Passing to the limit we get  $\frac{\partial W^{\tilde{\beta}}}{\partial e_{\tilde{\beta}}}(x_0) \geq 0$  in contradiction with (3.43). So we get (3.46).

By the maximum principle, the definition of  $B_\delta$  and (3.46) we get  $W^{\tilde{\beta}+\varepsilon} \leq 0$ , under both assumptions  $*$ ) and  $**$ ), also in  $B_{R_1} \cap \Omega_{\tilde{\beta}+\varepsilon} \cap B_\delta$  and hence (3.45) holds.

If  $\Omega = \mathbb{R}^N$  by the same argument, directly from (3.43) we get (3.45).

Now we want to prove that

$$(3.47) \quad W^{\tilde{\beta}+\varepsilon} \leq 0 \quad \text{in } \Omega_{\tilde{\beta}+\varepsilon} \quad \text{for all } \varepsilon \in [0, \varepsilon_1].$$

Because  $R_0 < R_1$ , by (3.45) the function  $v := (W^{\tilde{\beta}+\varepsilon})^+ \chi_{\Omega_{\tilde{\beta}+\varepsilon}}$  has its support strictly contained in  $\Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}$ . We claim that

$$(3.48) \quad v \equiv 0.$$

We first consider the case where assumption  $*$ ) is satisfied. Let  $\phi \in C_c^\infty(\Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}, \mathbb{R}^m)$ . By (3.44), the bilinear form  $P_U$ , defined in (2.30) defines a (semidefinite) scalar product on  $C_c^1(\Omega \setminus B_{R_0}, \mathbb{R}^m)$  and also on  $C_c^1(\Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}, \mathbb{R}^m)$ , and the

corresponding Cauchy-Schwarz-inequality yields

$$\left( P_U(\psi, \varphi, \Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}) \right)^2 \leq P_U(\psi, \psi, \Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}) P_U(\varphi, \varphi, \Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0})$$

for all  $\psi, \varphi$  in  $H_0^1(\Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}, \mathbb{R}^m)$  that vanish a.e. outside a bounded set. Consequently, we obtain

$$(3.49) \quad \left( P_U(v_R, \phi, \Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}) \right)^2 \leq Q_U(v_R, \Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}) Q_U(\phi, \Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}) \quad \text{for } R > 0,$$

where  $v_R = v\xi_R$  and  $\xi_R$  is defined in (2.31). The function  $v$  is nonnegative in  $\Omega_{\tilde{\beta}+\varepsilon}$ , so we are in position to apply Lemma 2.8, part *i*), getting from (2.32)

$$\limsup_{R \rightarrow \infty} Q_U(v_R, \Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}) \leq 0.$$

Combining this with (3.49), we find that

$$\begin{aligned} \int_{\Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}} \sum_{i=1}^m \nabla v_i \cdot \nabla \phi_i - \frac{1}{2} \sum_{i,j=1}^m \left( \frac{\partial f_i}{\partial u_j}(|x|, U) + \frac{\partial f_j}{\partial u_i}(|x|, U) \right) v_j \phi_i \, dx \\ = \lim_{R \rightarrow \infty} P_U(v_R, \phi, \Omega \setminus B_{R_0}) = 0. \end{aligned}$$

Since  $\phi \in C_c^\infty(\Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}, \mathbb{R}^m)$  was chosen arbitrarily, we conclude that  $v$  is a weak solution of the linear symmetric system

$$-\Delta v_i - \frac{1}{2} \sum_{j=1}^m \left( \frac{\partial f_i}{\partial u_j}(|x|, U) + \frac{\partial f_j}{\partial u_i}(|x|, U) \right) v_j = 0 \quad \text{in } \Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}$$

for  $i = 1, \dots, m$ . Then however  $v = (W^{\tilde{\beta}+\varepsilon})^+ \chi_{\Omega_{\tilde{\beta}+\varepsilon}} \equiv 0$  by the unique continuation principle, since  $(W^{\tilde{\beta}+\varepsilon})^+ \equiv 0$  in  $B_{R_1} \cap \Omega_{\tilde{\beta}+\varepsilon}$  by (3.45) and  $R_1 > R_0$ . Hence (3.48) holds.

Next we consider the case where hypothesis **\*\***) holds. Since every function  $\tau \in C_c^1(\Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}, \mathbb{R}^m)$  can be extended to an odd function  $\tilde{\tau} \in C_c^1(\Omega \setminus B_{R_0}, \mathbb{R}^m)$  with respect to the reflection at  $T_{\tilde{\beta}+\varepsilon}$ , we have by (3.44):

$$Q_{e_{\tilde{\beta}+\varepsilon}s}(\tau, \Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}) = \frac{1}{2} Q_U(\tilde{\tau}, \Omega \setminus B_{R_0}) \geq 0$$

for all  $\tau \in C_c^1(\Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}, \mathbb{R}^m)$ . Hence the bilinear form  $P_{e_{\tilde{\beta}+\varepsilon}s}$  associated with  $Q_{e_{\tilde{\beta}+\varepsilon}s}$  defines a (semidefinite) scalar product on  $C_c^1(\Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}, \mathbb{R}^m)$ , and the corresponding Cauchy-Schwarz-inequality reads

$$(3.50) \quad \begin{aligned} \left( P_{e_{\tilde{\beta}+\varepsilon}s}(v_R, \phi, \Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}) \right)^2 &\leq Q_{e_{\tilde{\beta}+\varepsilon}s}(\phi, \Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}) Q_{e_{\tilde{\beta}+\varepsilon}s}(v_R, \Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}) \\ &\leq Q_{e_{\tilde{\beta}+\varepsilon}s}(\phi, \Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}) \limsup_{R \rightarrow +\infty} Q_{e_{\tilde{\beta}+\varepsilon}s}(v_R, \Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}) \end{aligned}$$

for  $v_R := v\xi_R$  and for any  $\phi \in C_c^1(\Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}, \mathbb{R}^m)$ . Using *ii*) of Lemma 2.8 with  $D = \Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}$  then we have

$$\limsup_{R \rightarrow +\infty} Q_{e_{\tilde{\beta}+\varepsilon}s}(v_R, \Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}) \leq 0.$$

Then, from (3.50) it follows that

$$\int_{\Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}} \sum_{i=1}^m \nabla v_i \cdot \nabla \phi_i + \frac{1}{2} \sum_{i,j=1}^m (b_{ij}^{e_{\tilde{\beta}+\varepsilon}^s}(x) + b_{ji}^{e_{\tilde{\beta}+\varepsilon}^s}(x)) v_j \phi_i dx = 0$$

for any  $\phi \in C_c^1(\Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}, \mathbb{R}^m)$ . Then  $(W^{\tilde{\beta}+\varepsilon})^+$  is a solution of the linear symmetric system

$$-\Delta w_i^+ + \frac{1}{2} \sum_{j=1}^m (b_{ij}^{e_{\tilde{\beta}+\varepsilon}^s}(x) + b_{ji}^{e_{\tilde{\beta}+\varepsilon}^s}(x)) w_j^+ = 0, \text{ in } \Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}$$

for  $i = 1, \dots, m$ . Then, the unique continuation principle, implies that  $v \equiv 0$  in  $\Omega_{\tilde{\beta}+\varepsilon} \setminus B_{R_0}$  and (3.48) holds also in this case.

As a consequence of (3.48), we have got (3.47). Then the definition of  $\tilde{\beta}$  implies that  $W^{\tilde{\beta}} \equiv 0$  in  $\Omega_{\tilde{\beta}}$  and this gives the claim.  $\square$

#### 4. Proofs of Theorem 1.1 and Theorem 1.2

**Proposition 4.1.** *Let  $U$  be a solution of (1.1) and (1.2) with Morse index  $\mu(U) \leq N$  and assume that the system is fully coupled along  $U$  in  $\Omega(e)$  for any  $e \in S^{N-1}$ . Then there exists a direction  $e \in S^{N-1}$  such that*

$$(4.51) \quad Q_U(\psi, \Omega(e)) \geq 0 \quad \text{for any } \psi \in C_c^1(\Omega(e), \mathbb{R}^m).$$

*Proof.* The case  $\mu(U) < 2$  is immediate. Indeed at least one among  $Q_U(-, \Omega(e))$  and  $Q_U(-, \Omega(-e))$  should be positive semidefinite, otherwise we would obtain a 2-dimensional subspace of  $C_c^1(\Omega, \mathbb{R}^m)$  where the quadratic form  $Q_U(-, \Omega)$  is negative definite, contradicting the assumption of Morse index less than 2.

So let us assume  $2 \leq j := \mu(U) \leq N$ . By definition,  $j$  is the maximal dimension of a subspace  $X_j := \text{span}\{\Psi_1, \dots, \Psi_j\} \subset C_c^1(\Omega, \mathbb{R}^m)$  such that  $Q_U(\psi, \Omega) < 0$  for any  $\psi \in X_j \setminus \{0\}$ . We take a ball  $B_\rho$  with radius  $\rho > 0$  sufficiently large to contain the supports of all  $\Psi_i$ ,  $i = 1, \dots, j$ . For  $R \geq \rho$ , in the domain  $\Omega \cap B_R$  the linearized operator  $L_U$  defined in (2.28) has exactly  $j$  negative symmetric eigenvalues,  $\lambda_1^s(L_U, \Omega \cap B_R) < \lambda_2^s(L_U, \Omega \cap B_R) \leq \dots \leq \lambda_j^s(L_U, \Omega \cap B_R)$  with respect to Dirichlet boundary conditions, and  $\lambda_{j+1}^s(L_U, \Omega \cap B_R) \geq 0$ . See Section 2 for the definition of the symmetric eigenvalues and [2] for their variational characterization. Now assume, arguing by contradiction, that for any  $e \in S^{N-1}$  (4.51) does not hold. Then, we can apply Lemma 2.3 to the linear operator  $L_U$  and we can find a  $\tilde{R} > 0$  such that, for any  $R \geq \tilde{R}$  and for any  $e \in S^{N-1}$  the first symmetric eigenvalue  $\lambda_1^s(L_U, \Omega(e) \cap B_R)$  of the linearized operator in  $\Omega(e) \cap B_R$ , with zero Dirichlet boundary condition is negative.

We can take  $R \geq \max\{\rho, \tilde{R}, \bar{R}\}$  where  $\bar{R}$  is as in Lemma 2.6. In this way we have that the linearized system, defined in (2.28), is fully coupled in  $\Omega(e) \cap B_R$  for any  $e \in S^{N-1}$  and the same holds for the symmetric system associated with the linearized operator in  $\Omega(e) \cap B_R$ . We denote by  $\Phi_e$  the first positive  $L^2$ -normalized eigenfunction of the symmetric system  $-\Delta - \frac{1}{2}(J_F(|x|, U(x)) + J_F^t(|x|, U(x)))$  in  $\Omega(e) \cap B_R$  (we observe that  $\Phi_e$  is uniquely determined since the system is fully coupled in  $\Omega(e) \cap B_R$  for any  $e \in S^{N-1}$ ) and by  $\Phi_1, \dots, \Phi_j$  the mutually orthogonal

eigenfunctions corresponding to the  $j$  negative symmetric eigenvalues of  $L_U$  in  $\Omega \cap B_R$ . Define

$$\Psi_e(x) = \begin{cases} \left( \frac{(\Phi_{-e}, \Phi_1)_{L^2(\Omega(-e))}}{(\Phi_e, \Phi_1)_{L^2(\Omega(e))}} \right)^{\frac{1}{2}} \Phi_e(x) & \text{if } x \in \Omega(e) \cap B_R \\ - \left( \frac{(\Phi_{-e}, \Phi_1)_{L^2(\Omega(-e))}}{(\Phi_e, \Phi_1)_{L^2(\Omega(e))}} \right)^{\frac{1}{2}} \Phi_{-e}(x) & \text{if } x \in \Omega(-e) \cap B_R \end{cases}$$

where  $(-, -)_{L^2(D)}$  denotes the scalar product in  $L^2(D, \mathbb{R}^m)$ . The mapping  $e \mapsto \Psi_e$  is a continuous odd function from  $S^{N-1}$  to  $H_0^1(\Omega, \mathbb{R}^m)$  and by construction  $(\Psi_e, \Phi_1)_{L^2(\Omega \cap B_R)} = 0$ . Therefore the mapping  $h : S^{N-1} \rightarrow \mathbb{R}^{j-1}$  defined by

$$h(e) = ((\Psi_e, \Phi_2)_{L^2(\Omega \cap B_R)}, \dots, (\Psi_e, \Phi_j)_{L^2(\Omega \cap B_R)})$$

is an odd continuous mapping, and since  $j-1 < N$  by the Borsuk- Ulam Theorem it must have a zero. This means that there exists a direction  $e \in S^{N-1}$  such that  $\Psi_e$  is orthogonal to all the eigenfunctions  $\Phi_1, \dots, \Phi_j$  in  $L^2(\Omega \cap B_R, \mathbb{R}^m)$ . Since  $\mu(U) = j$  this implies that  $\frac{Q_U(\Psi_e, \Omega \cap B_R)}{(\Psi_e, \Psi_e)_{L^2(\Omega \cap B_R)}} \geq \lambda_{j+1}^s(L_U, \Omega \cap B_R) \geq 0$  against the fact that

$$\begin{aligned} Q_U(\Psi_e, \Omega \cap B_R) &= \left( \frac{(\Phi_{-e}, \Phi_1)_{L^2(\Omega(-e))}}{(\Phi_e, \Phi_1)_{L^2(\Omega(e))}} \right) \lambda_1^s(L_U, \Omega(e) \cap B_R) \\ &\quad + \left( \frac{(\Phi_{-e}, \Phi_1)_{L^2(\Omega(-e))}}{(\Phi_e, \Phi_1)_{L^2(\Omega(e))}} \right) \lambda_1^s(L_U, \Omega(-e) \cap B_R) < 0 \end{aligned}$$

by construction. The contradiction proves the assertion.  $\square$

Now we are in position to prove Theorem 1.1.

**Proof of Theorem 1.1.** By Proposition 4.1 we have that there exists a direction  $e \in S^{N-1}$  such that (4.51) holds. Hypothesis i) of Theorem 1.1 implies that the system (1.1) is fully coupled along  $U$  also in  $\Omega$  and so, if  $W^e \equiv 0$  in  $\Omega(e)$ , we immediately get the foliated Schwarz symmetry of  $U$ , by Proposition 3.7.

If instead  $W^e \not\equiv 0$  we show that  $W^e$  is either strictly positive or strictly negative in  $\Omega(e)$  so that, by Proposition 3.9 we again get the assertion.

From (4.51) we have that  $P_U(\psi, \phi, \Omega(e))$  is a semidefinite scalar product on  $C_c^1(\Omega(e), \mathbb{R}^m)$ . Consequently, using the Cauchy-Schwarz-inequality we obtain:

$$(4.52) \quad (P_U(v_R, \phi, \Omega(e)))^2 \leq Q_U(v_R, \Omega(e)) Q_U(\phi, \Omega(e))$$

for any  $\phi \in C_c^1(\Omega(e), \mathbb{R}^m)$ , where  $v_R := (W^e)^+ \chi_{\Omega(e)} \xi_R$  and  $\xi_R$  is a cut-off function as defined in (2.31). By Lemma 2.8 we have  $\limsup_{R \rightarrow +\infty} Q_U(v_R, \Omega(e)) \leq 0$ . Combining this with (4.52) and passing to the limit as  $R \rightarrow +\infty$  we have

$$\begin{aligned} \int_{\Omega(e)} \nabla(W^e)^+ \cdot \nabla \phi - \frac{1}{2} \sum_{i,j=1}^m \left( \frac{\partial f_i}{\partial u_j}(|x|, U) + \frac{\partial f_j}{\partial u_i}(|x|, U) \right) w_j^+ \phi_i dx \\ = \lim_{R \rightarrow +\infty} P_U(v_R, \phi, \Omega(e)) = 0. \end{aligned}$$

Since  $\phi \in C_c^1(\Omega(e), \mathbb{R}^m)$  was chosen arbitrarily, we conclude that  $(W^e)^+$  is a solution of the system

$$(4.53) \quad -\Delta w_i^+ - \frac{1}{2} \sum_{j=1}^m \left( \frac{\partial f_i}{\partial u_j}(|x|, U) + \frac{\partial f_j}{\partial u_i}(|x|, U) \right) w_j^+ = 0 \quad \text{in } \Omega(e)$$

for  $i = 1, \dots, m$ . Now, since  $(W^e)^+ \geq 0$  in  $\Omega(e)$  and the linear system (4.53) is fully coupled in  $\Omega(e)$ , the Strong Maximum Principle implies that either  $(W^e)^+ \equiv 0$  or  $(W^e)^+ > 0$  in  $\Omega(e)$ . In any case  $W^e$  is strictly positive or strictly negative in  $\Omega(e)$ .  $\square$

**Proposition 4.2.** *Let  $U$  be a solution of (1.1) and (1.2) with Morse index  $\mu(U) = j \leq N - 1$  and assume that the system (1.1) is fully coupled along  $U$  in  $\Omega$ . Then there exists a direction  $e \in S^{N-1}$  such that*

$$(4.54) \quad Q_{es}(\psi, \Omega(e)) := \int_{\Omega(e)} |\nabla \psi|^2 + \sum_{i,j=1}^m b_{ij}^{es}(x) \psi^i \psi^j dx \geq 0$$

for any  $\psi \in C_c^1(\Omega(e), \mathbb{R}^m)$ .

*Proof.* Assume, arguing by contradiction, that for any  $e \in S^{N-1}$  (4.54) does not hold. Then we can apply Lemma 2.3 to the linear operator  $L^{es}$ , defined in (2.23), and we can find  $\tilde{R} > 0$  such that, for any  $R \geq \tilde{R}$  and for any  $e \in S^{N-1}$  the first symmetric eigenvalue  $\lambda_1^s(L^{es}, \Omega(e) \cap B_R)$  of the linear operator  $L^{es}$  in  $\Omega(e) \cap B_R$  with Dirichlet boundary conditions is negative.

By definition  $j$  is the maximal dimension of a subspace  $X_j := \text{span} \langle \Psi_1, \dots, \Psi_j \rangle \subset C_c^1(\Omega, \mathbb{R}^m)$  such that  $Q_U(\psi, \Omega) < 0$  for any  $\psi \in X_j \setminus \{0\}$ . We take a ball  $B_\rho$  with radius  $\rho > 0$  sufficiently large to contain the supports of all  $\Psi_i$ ,  $i = 1, \dots, j$ . For  $R \geq \rho$ , in the domain  $\Omega \cap B_R$  the linearized operator  $L_U$  has exactly  $j$  negative symmetric eigenvalues and  $\lambda_{j+1}^s(L_U, B_R \cap \Omega) \geq 0$ .

We take  $R \geq \max\{\rho, \tilde{R}, \bar{R}\}$ , where  $\bar{R}$  is as defined in Remark 2.7. In this way we have that the linear system  $-\Delta + B^{es}(x)$  defined in (2.23), is fully coupled in  $\Omega(e) \cap B_R$  for any  $e \in S^{N-1}$  and the same holds for the symmetric system associated with the linear operator  $L^{es}$  in  $\Omega(e) \cap B_R$ . Moreover the system (1.1) is fully coupled along  $U$  in  $\Omega \cap B_R$ .

We denote by  $g_e$  the first positive  $L^2$ -normalized eigenfunction corresponding to  $\lambda_1^s(L^{es}, \Omega(e) \cap B_R)$  (which is defined since the system  $-\Delta V + B^{es}(x)V$  is fully coupled in  $\Omega(e) \cap B_R$  for any  $e \in S^{N-1}$ ) and by  $\tilde{g}_e$  the odd extension to  $\Omega \cap B_R$ . Let  $\Phi_1, \dots, \Phi_j$  be the mutually orthogonal eigenfunctions corresponding to the  $j$  negative symmetric eigenvalues of  $L_U$  in  $\Omega \cap B_R$ . Let  $h : S^{N-1} \rightarrow \mathbb{R}^j$  be defined by

$$h(e) := ((\tilde{g}_e, \Phi_1)_{L^2(\Omega \cap B_R)}, \dots, (\tilde{g}_e, \Phi_j)_{L^2(\Omega \cap B_R)})$$

where  $(-, -)_{L^2(D)}$  is, as before, the usual scalar product in  $L^2(D, \mathbb{R}^m)$ .  $h$  is an odd and continuous mapping and, since  $j < N - 1$  it must have a zero by the Borsuk-Ulam Theorem. This means that there exists a direction  $e \in S^{N-1}$  such that  $\tilde{g}_e$  is orthogonal to all the eigenfunctions  $\Phi_1, \dots, \Phi_j$  in  $L^2(\Omega \cap B_R, \mathbb{R}^m)$ . Since  $\mu(U) = j$  this implies that

$$\frac{Q_U(\tilde{g}_e, \Omega \cap B_R)}{(\tilde{g}_e, \tilde{g}_e)_{L^2(\Omega \cap B_R)}} = \frac{\int_{\Omega \cap B_R} |\nabla \tilde{g}_e|^2 - \sum_{i,j=1}^m \frac{\partial f_i}{\partial u_j}(|x|, U) \tilde{g}_e^i \tilde{g}_e^j dx}{(\tilde{g}_e, \tilde{g}_e)_{L^2(\Omega \cap B_R)}} \geq \lambda_{j+1}^s(L_U, \Omega \cap B_R) \geq 0.$$

The symmetry properties of  $\tilde{g}_e$  imply that

$$\begin{aligned} \int_{\Omega} |\nabla \tilde{g}_e|^2 - \sum_{i,j=1}^m \frac{\partial f_i}{\partial u_j}(|x|, U) \tilde{g}_e^i \tilde{g}_e^j dx &= 2 \int_{\Omega(e) \cap B_R} |\nabla g_e|^2 + \sum_{i,j=1}^m b_{ij}^{es}(x) g_e^i g_e^j dx \\ &= 2\lambda_1^s(L^{es}, \Omega(e) \cap B_R) < 0. \end{aligned}$$

The contradiction proves the assertion.  $\square$

**Proof of Theorem 1.2.** From Proposition 4.2 we have a direction  $e \in S^{N-1}$  such that  $Q_{es}(\psi, \Omega(e)) \geq 0$  for any  $\psi \in C_c^1(\Omega(e), \mathbb{R}^m)$ . If  $W^e \equiv 0$ , i.e. if  $U$  is symmetric with respect to  $T(e)$ , then  $Q_{es}(\psi, \Omega(e)) = Q_U(\psi, \Omega(e)) \geq 0$  for any  $\psi \in C_c^1(\Omega(e), \mathbb{R}^m)$ . The foliated Schwarz symmetry of  $U$  then follows from Proposition 3.7.

So assume  $W^e \not\equiv 0$ . By assumption the bilinear form  $P_{es}$ , defined in (2.25), defines a scalar product on  $C_c^1(\Omega(e), \mathbb{R}^m)$ , so, for  $v_R = (W^e)^+ \xi_R \in H_0^1(\Omega(e), \mathbb{R}^m)$  and for any  $\phi \in C_c^1(\Omega(e), \mathbb{R}^m)$ , we have

$$(4.55) \quad 0 \leq \left( P_{es}(v_R, \phi, \Omega(e)) \right)^2 \leq Q_{es}(v_R, \Omega(e)) Q_{es}(\phi, \Omega(e)).$$

Moreover, using (2.33) with  $D = \Omega(e)$ , we have

$$\limsup_{R \rightarrow +\infty} Q_{es}(v_R, \Omega(e)) \leq 0.$$

Passing to the limit in (4.55) we get

$$P_{es}((W^e)^+, \phi, \Omega(e)) = 0$$

for any  $\phi \in C_c^1(\Omega(e), \mathbb{R}^m)$ , so that  $(W^e)^+$  is a weak solution of

$$(4.56) \quad -\Delta(w_i)^+ + \frac{1}{2} \sum_{j=1}^m (b_{ij}^{es}(x) + b_{ji}^{es}(x))(w_j)^+ = 0 \quad \text{in } \Omega(e)$$

for  $i = 1, \dots, m$ . Since the system is fully coupled in  $\Omega(e)$ , the strong maximum principle implies that either  $(W^e)^+ > 0$  in  $\Omega(e)$  or  $(W^e)^+ \equiv 0$  in  $\Omega(e)$ . In any case  $W^e$  is strictly positive or strictly negative in  $\Omega(e)$  and the foliated Schwarz symmetry of  $U$  follows from Proposition 3.9.  $\square$

## 5. Other results

We prove the other theorems stated in Section 1.

*Proof of Theorem 1.3.* If  $U$  is a Morse index one solution for any direction  $e \in S^{N-1}$  at least one among  $Q_U(-, \Omega(e))$  and  $Q_U(-, \Omega(-e))$  should be positive semidefinite, otherwise we would obtain a 2-dimensional subspace of  $C_c^1(\Omega, \mathbb{R}^m)$  where the quadratic form  $Q_U(-, \Omega)$  is negative defined, contradicting the definition of Morse index 1.

By the proof of Theorem 1.1 and of Theorem 1.2 we can find a direction  $e \in S^{N-1}$  such that  $W^e \equiv 0$  in  $\Omega(e)$  or  $W^e > 0$  in  $\Omega(e)$ . In the second case, applying Proposition 3.9 we can find a direction  $e'$  such that  $W^{e'} \equiv 0$  in  $\Omega(e')$ . So, in any case, there exists a direction  $e$  such that  $U$  is symmetric with respect to the hyperplane  $T(e)$ . Thus, by symmetry,  $Q_U(\psi, \Omega(e)) = Q_U(\psi, \Omega(-e))$  for any  $\psi \in C_c^1(\Omega(e), \mathbb{R}^m)$  and  $Q_U$  is positive semidefinite in  $\Omega(e)$ .

After a rotation, we may assume that  $e = e_2 = (0, 1, \dots, 0)$  so that  $T(e) = \{x \in \mathbb{R}^N : x_2 = 0\}$  and we may introduce new (cylinder) coordinates  $(r, \theta, y_3, \dots, y_N)$  defined by the relations  $x = r[\cos \theta e_1 + \sin \theta e_2] + \sum_{i=3}^N y_i e_i$ .

Then the angular derivative  $U_\theta$  of  $U$  with respect to  $\theta$ , extended by zero at the origin if  $\Omega$  is a ball, satisfies the linearized system, i.e.

$$(5.57) \quad -\Delta U_\theta - J_F(|x|, U) U_\theta = 0 \quad \text{in } \Omega(e_2)$$



and it is zero on the boundary. Reasoning exactly as in the proof of Proposition 3.7 we have that  $U_\theta^+$  is a solution of

$$(5.58) \quad -\Delta U_\theta^+ - \frac{1}{2} (J_F(|x|, U) + J_F^t(|x|, U)) U_\theta^+ = 0 \quad \text{in } \Omega(e_2)$$

and this implies that  $U_\theta$  does not change sign in  $\Omega(e_2)$ . We can assume that  $U_\theta = U_\theta^+$ , and that  $U_\theta^+$  is a solution of the systems (5.57) and (5.58). Then  $J_F(|x|, U)U_\theta = \frac{1}{2} (J_F(|x|, U) + J_F^t(|x|, U)) U_\theta$ , i.e. (1.6) and if  $m = 2$ , since  $U_\theta$  is positive, we get (1.7). The case  $U_\theta = U_\theta^-$  can be handled in the same way. This proves the assertion if the hypothesis a) holds.

To prove the theorem under assumption b), we observe that the result follows if we can find a direction  $e$  such that  $W^e \equiv 0$  in  $\Omega(e)$  and  $Q_U(\psi, \Omega(e)) \geq 0$  for any  $\psi \in C_c^1(\Omega(e), \mathbb{R}^m)$ .

Following the proof of Theorem 1.2 we have a direction  $e \in S^{N-1}$  such that either  $W^e \equiv 0$  and  $Q_U(\psi, \Omega(e)) \geq 0$  or  $W^e > 0$  in  $\Omega(e)$ .

The second case cannot happen. Indeed the function  $W^e$  satisfies the system

$$(5.59) \quad -\Delta w_i + \sum_{j=1}^m b_{ij}^e(x) w_j = 0 \quad \text{in } \Omega(e)$$

where  $b_{ij}^e(x)$  are as in Lemma 2.4. Multiplying the  $i$ -th equation of (4.56) and (5.59) for  $w_i \xi_R$  ( $\xi_R$  is the usual cutoff function, see (2.31)), integrating in  $\Omega(e)$ , summing on  $i$  and subtracting, we get

$$\sum_{i,j=1}^m \int_{\Omega(e)} \left( b_{ij}^e(x) - \frac{1}{2} (b_{ij}^{es}(x) + b_{ji}^{es}(x)) \right) w_j w_i \xi_R dx = 0$$

equivalently

$$(5.60) \quad \sum_{i,j=1}^m \int_{\Omega(e)} (b_{ij}^e(x) - b_{ij}^{es}(x)) w_j w_i \xi_R dx = 0.$$

Since  $W^e > 0$ , Lemma 2.4 implies that  $b_{ij}^e(x) \geq b_{ij}^{es}(x)$  for any  $i, j = 1, \dots, m$ , and since  $w_i w_j \xi_R > 0$  in  $\Omega(e)$ , relation (5.60) gives  $b_{ij}^e(x) = b_{ij}^{es}(x)$  for any  $i, j = 1, \dots, m$ .

By the strict convexity of  $\frac{\partial f_{i_0}}{\partial u_{j_0}}$  we have that, if  $W^e > 0$  then  $b_{i_0, j_0}^e(x) > b_{i_0, j_0}^{es}(x)$  getting a contradiction. Therefore the only possible case is  $W^e \equiv 0$  in  $\Omega(e)$  and  $Q_U(\psi, \Omega(e)) \geq 0$ . Then the result follows as in the previous case.  $\square$

*Proof of Theorem 1.4.* We choose, as before, the cylindrical coordinates with respect to the plane  $x_1 x_2$ , i.e.  $(r, \theta, \tilde{x})$ . Again the derivative  $U_\theta$  satisfies the linearized system  $-\Delta U_\theta - J_F(|x|, U)U_\theta = 0$  in  $\Omega$  and, in the case  $\Omega = \mathbb{R}^N \setminus B_R(0)$ , also the boundary conditions  $U_\theta = 0$  on  $\partial\Omega$ . By the stability assumption, we can proceed as in the proof of Proposition 3.7, with  $\Omega$  in place of  $\Omega(e)$ , to show that  $U_\theta$  does not change sign in  $\Omega$ . Since  $U_\theta$  is  $2\pi$ -periodic this is impossible and therefore  $U_\theta \equiv 0$ . By the arbitrariness of  $x_1, x_2$  we conclude that  $U$  is radial.

Moreover, if  $\Omega = \mathbb{R}^N$  and  $F$  does not depend on  $|x|$ , then for every  $t$  the translated function  $U(x + t)$  is also a stable solution of (1.1), therefore it is radial by the argument above. This however is not possible unless  $U$  is constant.  $\square$

*Proof of Theorem 1.6.* Suppose by contradiction that (1.1) admits a sign changing solution  $U$  on  $\mathbb{R}^N$  that satisfies the assumptions of Theorem 1.3 and such that

$\lim_{|x| \rightarrow \infty} U(x) = 0$ . Since we can apply Theorems 1.1 and 1.2 then  $U$  is foliated Schwarz symmetric. By a rotation of coordinates, we may assume that  $p = e_N$  in the definition of foliated Schwarz symmetry, so that  $U$  is axially symmetric with respect to the axis  $\mathbb{R}e_N$  and nonincreasing in the angle  $\theta = \arccos \frac{x_N}{|x|}$ . By the proofs of Theorem 1.3 we get a direction  $e \in S^{N-1}$  such that  $U$  is symmetric with respect to  $T(e)$  and

$$(5.61) \quad \inf_{\psi \in C_c^1(\Omega(e))} Q_U(\psi, \Omega(e)) \geq 0.$$

We may assume that  $e = e_1$  in (5.61). Indeed, this is clearly possible if  $U$  is radial. Moreover, if  $U$  is nonradial, then  $U$  is strictly decreasing in the angle  $\theta$ , therefore the symmetry hyperplanes of  $U$  are precisely the ones containing  $e_N$ , and for each one of them the infimum in (5.61) takes the same value.

We now consider the derivative  $\frac{\partial U}{\partial x_1}$  which, by regularity theory, (see [9] Lemma 6.1) belongs to  $H^2(\Omega(e_1))$  and satisfies the linearized system

$$(5.62) \quad -\Delta \left( \frac{\partial U}{\partial x_1} \right) - J_F(U(x)) \left( \frac{\partial U}{\partial x_1} \right) = 0 \quad \text{in } \Omega(e_1).$$

Moreover, because of the symmetry with respect to  $T(e_1)$  we have

$$\frac{\partial U}{\partial x_1} = 0 \quad \text{on } \partial\Omega(e_1)$$

so that  $\frac{\partial U}{\partial x_1} \in H_0^1(\Omega(e_1))$ . From (5.61) we have that  $P_U$  defines a (semidefinite) scalar product on  $C_c^1(\Omega(e_1), \mathbb{R}^m)$  and the corresponding Cauchy-Schwarz-inequality reads

$$(5.63) \quad (P_U(v_R, \phi, \Omega(e_1)))^2 \leq Q_U(\phi, \Omega(e_1)) \limsup_{R \rightarrow +\infty} Q_U(v_R, \Omega(e_1))$$

if  $v_R = \left( \frac{\partial U}{\partial x_1} \right)^+ \xi_R$ , with  $\xi_R$  being the usual cutoff function. Reasoning exactly as in the proof of Proposition 3.7, with  $\left( \frac{\partial U}{\partial x_1} \right)^+$  instead of  $U_\theta^+$ , we get

$$Q_U(v_R, \Omega(e_1)) \leq \int_{\Omega(e_1)} \left| \left( \frac{\partial U}{\partial x_1} \right)^+ \right|^2 |\nabla \xi_R|^2 dx \leq \frac{1}{R^2} \int_{B_{2R} \setminus B_R} |\nabla U|^2 dx$$

so that

$$\limsup_{R \rightarrow +\infty} Q_U(v_R, \Omega(e_1)) \leq 0.$$

Then (5.63) implies, as before, that  $\left( \frac{\partial U}{\partial x_1} \right)^+$  is a weak solution of the system

$$(5.64) \quad -\Delta \left( \frac{\partial U}{\partial x_1} \right)^+ - \frac{1}{2} (J_F(U(x)) + J_F^t(U(x))) \left( \frac{\partial U}{\partial x_1} \right)^+ = 0 \quad \text{in } \Omega(e_1).$$

Since this system is fully coupled in  $\Omega(e_1)$  the Strong Maximum principle implies that either  $\left( \frac{\partial U}{\partial x_1} \right)^+ \equiv 0$  in  $\Omega(e_1)$  or  $\frac{\partial U}{\partial x_1} > 0$  in  $\Omega(e_1)$ . In any case  $\frac{\partial U}{\partial x_1}$  does not change sign in  $\Omega(e_1)$  which is impossible because  $U$  changes sign in  $\Omega(e_1)$  and  $\lim_{|x| \rightarrow \infty} U(x) = 0$ . This contradiction proves the assertion.  $\square$

*Proof of Theorem 1.7.* Suppose by contradiction that (1.1) and (1.2) admit a positive solution  $U$  in  $\mathbb{R}^N \setminus B$  that satisfies the assumptions of Theorem 1.3 and such that  $\lim_{|x| \rightarrow \infty} U(x) = 0$ . As in the proof of Theorem 1.6 we may assume that  $U$  is symmetric with respect to  $T(e_1)$  and that (5.61) holds for  $e = e_1$ . Then the derivative  $\frac{\partial U}{\partial x_1}$  satisfies the system

$$(5.65) \quad \begin{cases} -\Delta \left( \frac{\partial U}{\partial x_1} \right) - J_F(U(x)) \left( \frac{\partial U}{\partial x_1} \right) = 0 & \text{in } \Omega(e_1) \\ \frac{\partial U}{\partial x_1} = 0 & \text{on } T(e_1) \cap \bar{\Omega} \\ \frac{\partial U}{\partial x_1} \geq 0 & \text{on } \partial B \cap \bar{\Omega}(e_1) \end{cases}$$

Multiplying by  $\left( \frac{\partial U}{\partial x_1} \right)^- \xi_R^2$ , integrating over  $\Omega(e_1)$  and using the cooperativeness of (1.1), we get

$$-\sum_{i=1}^m \int_{\Omega(-e_1)} \nabla \left( \frac{\partial u_i}{\partial x_1} \right) \cdot \nabla \left[ \left( \frac{\partial u_i}{\partial x_1} \right)^- \xi_R^2 \right] dx \leq \sum_{i,j=1}^m \int_{\Omega(-e_1)} \frac{\partial f_i}{\partial u_j}(U(x)) \left( \frac{\partial u_j}{\partial x_1} \right)^- \left( \frac{\partial u_i}{\partial x_1} \right)^- \xi_R^2 dx.$$

Then, as in the proof of the previous theorem, we get that  $\left( \frac{\partial U}{\partial x_1} \right)^-$  is a solution of the symmetric system (5.65) and hence  $\left( \frac{\partial U}{\partial x_1} \right)^- \equiv 0$  by the boundary conditions. Then  $\frac{\partial U}{\partial x_1} \geq 0$  in  $\Omega(e_1)$  contradicting the fact that  $U = 0$  on  $\partial B$  and  $U(x) \rightarrow 0$  as  $|x| \rightarrow +\infty$ .  $\square$

*Proof of Theorem 1.8.* The proof follows as in the case of Theorems 1.6 and 1.7 once we get, as in the proof of Theorem 1.3, the existence of a direction  $e \in S^{N-1}$  such that  $U$  is symmetric with respect to  $T(e)$  and that (5.61) holds.

To get (5.61) we need the following fact: if we have a direction  $e \in S^{N-1}$  such that  $W^e$  is either strictly positive or strictly negative in  $\Omega(e)$  and the system is of gradient type, then

$$(5.66) \quad Q_e(\psi, \Omega(e)) \geq 0 \quad \text{for any } \psi \in C_c^1(\Omega(e), \mathbb{R}^m).$$

This fact is a generalization of Lemma 2.1 in [9] and follows in a similar way.

Now, starting from the proof of Theorem 1.1 (or of Theorem 1.2) we get a direction  $e \in S^{N-1}$  such that  $Q_U(\psi, \Omega(e)) \geq 0$  ( $Q_{es}(\psi, \Omega(e)) \geq 0$  respectively) for any  $\psi \in C_c^1(\Omega(e), \mathbb{R}^m)$ . If  $W^e \equiv 0$  in  $\Omega(e)$  then (5.61) is satisfied ( $Q_{es}(\psi, \Omega(e)) = Q_U(\psi, \Omega(e))$ , by the symmetry) and we are done. If, else,  $W^e \not\equiv 0$  we have, as in the proof of Theorem 1.1 (Theorem 1.2 respectively) that  $W^e$  is either strictly positive or strictly negative in  $\Omega(e)$ .

Then, applying the rotating plane method, see Proposition 3.9, we get, using the same notations, the existence of  $\tilde{\beta} > 0$  such that  $W^{\tilde{\beta}} \equiv 0$  in  $\Omega(\tilde{\beta})$  and  $W^{\tilde{\beta}} < 0$  in  $\Omega(\tilde{\beta})$  for any  $\beta \in [0, \tilde{\beta})$ . This means, using (5.66), that  $Q_e(\psi, \Omega(\beta)) \geq 0$  for any  $\beta \in [0, \tilde{\beta})$  and, passing to the limit,  $Q_e(\psi, \Omega(\tilde{\beta})) \geq 0$ . The symmetry of  $U$  with respect to  $T(\tilde{\beta})$  then implies that  $Q_e(\psi, \Omega(\tilde{\beta})) = Q_U(\psi, \Omega(\tilde{\beta})) \geq 0$  and (5.61) is satisfied concluding the proof.  $\square$

## REFERENCES

- [1] J. Busca, B. Sirakov *Symmetry results for semilinear elliptic systems in the whole space*, J. Differential Equations 163 (2000), no. 1, 41-56.
- [2] L. Damascelli, F. Pacella *Symmetry results for cooperative elliptic systems via linearization*, Preprint, arXiv:1206.3926.

- [3] L. Damascelli, F. Gladiali, F. Pacella *A symmetry results for semilinear cooperative elliptic systems*, Recent trends in Nonlinear P.D.E., AMS -Contemporary Mathematics Series (in press), arXiv:1209.5581.
- [4] D.G. de Figueiredo, J. Yang *Nonlinear Anal.* 33 (1998), no. 3, 211234. *Decay, symmetry and existence of solutions of semilinear elliptic systems*, *Nonlinear Anal.* 33 (1998), no. 3, 211-234.
- [5] D.G. de Figueiredo *Semilinear Elliptic Systems: existence, multiplicity, symmetry of solutions*, Handbook of Differential Equations: stationary partial differential equations. Vol. V Eq. 2008, 1-48.
- [6] D.G. de Figueiredo, E. Mitidieri *Maximum principles for linear elliptic systems*, Rend. Inst. Mat. Univ. Trieste, 1992, pp. 36-66.
- [7] A. Farina *On the classification of solutions of the Lane-Emden equation on unbounded domains of  $\mathbb{R}^N$* , J. Math. Pures Appl. (9) 87 (5), 2007, pp. 537-561.
- [8] M. Fazly, N. Ghoussoub *On the Hénon-Lane-Emden conjecture*, arXiv:1107.5611v2.
- [9] F. Gladiali, F. Pacella, T. Weth *Symmetry and nonexistence of low Morse index solutions in unbounded domains*, J. Math. Pures Appl. (9) 93 (5), 2010, pp. 536-558.
- [10] L. A. Maia, E. Montefusco, B. Pellacci *Positive solutions for a weakly coupled nonlinear Schrodinger system*, J. Differential Equations 229, 2006, pp. 743-767.
- [11] F. Pacella *Symmetry results for solutions of semilinear elliptic equations with convex nonlinearities*, J. Funct. Anal. 192, 2002, pp. 271-282.
- [12] F. Pacella, M. Ramaswamy *Symmetry of solutions of elliptic equations via maximum principles*, Handb. Differ. Equ.: stationary partial differential equations. Vol. VI, 269-312, Elsevier/North-Holland, Amsterdam, 2008.
- [13] F. Pacella, T. Weth, *Symmetry of solutions to semilinear elliptic equations via Morse index*, Proc. Amer. Math. Soc. 135 (6), 2007, pp. 1753-1762.
- [14] A. Saldana, T. Weth, *Asymptotic axial symmetry of solutions of parabolic equations in bounded radial domains*, J. Evol. Equ. 12 (3), 2012, pp. 697-712.
- [15] B. Sirakov, *Some estimates and maximum principles for weakly coupled systems of elliptic PDE*, Nonlinear Analysis 70 (8) , 2009, pp. 3039-3046.
- [16] T. Weth, *Symmetry of solutions to variational problems for nonlinear elliptic equations via reflection methods*, Jahresber. Dtsch. Math.-Ver. 112 (3), 2010, pp 119-158.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA " TOR VERGATA " - VIA DELLA RICERCA SCIENTIFICA 1 - 00173 ROMA - ITALY.

*E-mail address:* damascel@mat.uniroma2.it

DIPARTIMENTO POLCOMING-MATEMATICA E FISICA, UNIVERSITÀ DI SASSARI - VIA PIANDANNA 4, 07100 SASSARI - ITALY.

*E-mail address:* fgladiali@uniss.it

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA LA SAPIENZA - P.LE A. MORO 2 - 00185 ROMA - ITALY.

*E-mail address:* pacella@mat.uniroma1.it